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MATHEMATICS

On Certain Axiomatics for the Operational Calculus

by

R. BITTNER

Presented by S. MAZUR on October 28, 1958

1. Introduction

S. Bellert [1] has given the following axiomatics for the operational calculus:

Suppose we are given an endomorphism T of a linear space X over a field C of complex numbers satisfying the condition

$$\sum_{n=0}^{N} a_n T^n x \neq 0 \quad \text{for} \quad a_N \neq 0 \quad \text{and} \quad x \neq 0, \quad a_i \in C, \quad x \in X.$$

Then the ring of endomorphisms $\sum_{n=0}^{N} a_n T^n$ can be extended to the field of elements $\dfrac{a_0 + a_1 T + \ldots + a_n T^n}{b_0 + b_1 T + \ldots + b_m T^m}$, which gives in particular an operator $p = \dfrac{1}{T}$. Different interpretations of the space X and the endomorphism T correspond to the use of the Heaviside method in ordinary differential and difference equations with constant coefficients and in Euler equations, making it also possible to solve difference-differential equations with constant coefficients.

This paper is an attempt to axiomatize the operational calculus on the basis of the notion of limit condition and of the correct posing of the problem.

2. Algebraical equations correctly posed

Suppose we are given two groups, G_1 and G_2, and a homomorphism S from G_1 into G_2. We consider the equation $Sx = f$, where $x \in G_1$, $f \in G_2$. The general solution of this equation is of the form $x = a \cdot Tf = Tf \cdot b$, where T is an isomorphism with the property $STf = f$ for $f \in G_1$.

Thus, in order to determine the solution we must know either the element a or the element b. The condition $\gamma_L x = a$ will be called the left limit condition and the condition $\gamma_p x = b$ the right limit condition.

The formula $\gamma_L x = x\,(TSx)^{-1}$ gives

THEOREM 1. *The transformation γx of group G_1 into the kernel $J(S)$ of homomorphism S defines the left limit condition if and only if $\gamma e = e$, $\gamma x = e = \gamma y \supset \gamma(xy) = e$, where e is the unity of group G_1, $\gamma(ax) = a \cdot \gamma x$ for $a \in J(S)$, $x \in G_1$.*

A similar theorem can be proved for the right limit condition. In the rest of sec. **2** we shall speak of the left limit condition only.

The transformation $\gamma_L x$ is not a homomorphism because $\gamma_L(xy) = x \cdot \gamma_L y \cdot x^{-1} \cdot \gamma_L x$. In the case of an Abelian group, the left limit condition is identical with the right one and the transformation $\gamma_L x = \gamma_p x = \gamma x$ defines a homomorphism.

Now, let $G_2 \subset G_1$, i. e. let S be an endomorphism. We can then consider the equation $S^n x = f$. It can be proved by induction that

(1) if $S^n x = f$, then $x = c_0 \cdot Tc_1 \cdot T^2 c_2 \ldots T^{n-1} c_{n-1} \cdot T^n f$,

where $c_i = \gamma_L S^i x$, $i = 0, 1, \ldots, n-1$.

Further, let $A_n, A_{n-1}, \ldots, A_0$ be arbitrary endomorphisms of group G_1 commutative with the isomorphism T, i. e. having the property $TA_k x = A_k Tx$ for $x \in G_1$, $k = 0, 1, \ldots, n$.

Consider the equation

$$A_n S^n x \cdot A_{n-1} S^{n-1} x \cdot \ldots \cdot A_0 x = f. \tag{2}$$

By formula (1) this equation is equivalent to the equation

$$A_n[W_{n-1}^{-1}(x) \cdot x] \cdot A_{n-1} T\,[W_{n-2}^{-1}(x) \cdot x] \ldots A_1 T^{n-1}\,[W_0^{-1}(x) \cdot x] \cdot A_0 T^n(x) = T^n f,$$

where $W_k(x) = c_0 \cdot Tc_1 \ldots T^k c_k$, $c_l = \gamma_L S^l x$, $k, l = 0, 1, \ldots, n-1$.

Hence, we have

THEOREM 2. *The solution of Eq.* (2) *is reducible to the imposing of limit conditions upon the element x and to the iterations $S^k x$ up to and including the $(n-1)$-th one.*

The equation

$$\begin{aligned} A_n S^n x \cdot A_{n-1} S^{n-1} x \cdot \ldots \cdot A_0 x &= f, \quad f, x \in G_1, \\ \gamma_L S^k x &= c_k, \quad k = 0, 1, \ldots, n-1, \end{aligned} \tag{3}$$

where the endomorphisms $A_n, A_{n-1}, \ldots, A_0$ are commutative with the isomorphism T, will be called an equation posed correctly.

3. The algebra of operators on Abelian groups

Under the assumption that group G_1 is Abelian we shall write Eq. (3) in the form

$$A_n S^n x + A_{n-1} S^{n-1} x + \ldots + A_0 x = f, \qquad \gamma S^k x = c_k, \qquad f, x \in G_1, \quad k = 0, 1, \ldots, n-1. \tag{4}$$

Eq. (4) is equivalent to the equation

$$(A_n + A_{n-1} T + \ldots + A_0 T^n) x = \\ = A_n W_{n-1}(x) + A_{n-1} T W_{n-2}(x) + \ldots + A_1 T^{n-1} W_0(x) + T^n f,$$

where

$$W_k(x) = c_0 + T c_1 + \ldots + T^k c_k, \qquad k = 0, 1, \ldots, n-1$$

and thus it has the general form

$$V(T)x = \sum_{k=0}^{n-1} V_k(T) + T^n f, \tag{5}$$

where the coefficients in the polynomials $V(T), V_0(T), \ldots, V_n(T)$ of the isomorphism T are the endomorphisms $A_n, A_{n-1}, \ldots, A_0$ and we have $stV_k(T) < stV(T)$, $V(T) = A_n + A_{n-1} T + \ldots + A_0 T^n$.

If we assume that the endomorphisms A_i are commutative with the homomorphism S, then it can be proved that the polynomial $V(T)$ is decomposable into the product of polynomials of lower degrees with coefficients commutative with the operations S and T if and only if the solution of Eq. (4) is reducible to the solving of two equations of lower orders.

In particular, we can have the canonical decomposition

$$(S - R_1)^{k_1} (S - R_2)^{k_2} \ldots (S - R_m)^{k_m} x = f \\ \gamma S^k x = c_k, \quad k = 0, 1, \ldots, k_1 + \ldots + k_m - 1, \quad S c_k = 0, \quad f, x \in G_1, \tag{6}$$

where the endomorphisms $R_1, \ldots, R_m$ are commutative with the operations S and T.

If Eq. (6) is homogeneous $f = 0 = c_0 = \ldots = c_{n-1}$ and has a non-trivial solution, then there exists an element $x_0 \neq 0$ such that

$$(I - TR_1)^{k_1} (I - TR_2)^{k_2} \ldots (I - TR_m)^{k_m} x_0 = 0 \qquad (Ix = x \quad \text{for} \quad x \in G_1).$$

Endomorphisms R_i for which $(I - TR_i)x = 0 \supset x = 0$ will be called generators.

Consider now the set $\Pi[T]$ of polynomials of the form

$$W(T) = T^{k_0} (I - TR_1)^{k_1} \ldots (I - TR_m)^{k_m},$$

where $R_1, \ldots, R_m$ are generators and the exponents $k_0, k_1, \ldots, k_m$ are either natural numbers or zero.

The results $\frac{W_1}{W_2}x$, where $W_1, W_2 \in \Pi[T]$, $W_2 \neq 0$, $x \in G_1$, is a pair $\left[\frac{W_1}{W_2}, x\right]$ with the equality

$$\frac{W_1}{W_2}x = \frac{W_3}{W_4}y \equiv W_1W_4x = W_2W_3y\,, \tag{7}$$

and operations

$$\frac{W_1}{W_2}x + \frac{W_3}{W_4}y = \frac{I}{W_2W_4}(W_1W_4x + W_2W_3y)\,,$$

$$U\frac{W_1}{W_2}x = \frac{W_1}{W_2}Ux \quad \text{if} \quad UW(T) = W(T)U \quad \text{for every} \quad W(T) \tag{8}$$

$$\frac{W_1}{W_2}\cdot\frac{W_3}{W_4}x = \frac{W_1W_3}{W_2W_4}x\,.$$

The set of results $G(T)$ constitutes a group. The endomorphisms of that group are the operators $\frac{W_1(T)}{W_2(T)}$, $\frac{UW_1(T)}{W_2(T)}$, where $W_1(T), W_2(T) \in \Pi[T]$ and U is an endomorphism commutative with any polynomial $W(T) \in \Pi[T]$.

Just as a set of operators has a ring of polynomials $\left(\frac{W(T)}{I} = W(T)\right)$ isomorphically contained in it, so a group of results $G(T)$ has a group $G_1\left(x = \frac{I}{I}x\right)$ isomorphically contained in it.

By means of a group of results it is possible to solve Eq. (6). Let us use the general form (5) and let $p = 1/T$. We obtain

THEOREM 3. *Eq.* (6) *has the solution*

$$x = \sum_{k=0}^{n-1} \frac{U_k(p)}{U(p)}c_k + \frac{I}{U(p)}f\,, \tag{9}$$

where $st\,U_k(p) \leqslant st\,U(p)$, $U(p) = (p-R_1)^{k_1} \ldots (p-R_m)^{k_m}$.

In order to obtain a simpler form of the solution we can use the Heaviside algorithm consisting in a decomposition of rational operators $\frac{U_0(p)}{U(p)}, \ldots, \frac{U_{n-1}(p)}{U(p)}, \frac{I}{U(p)}$ into simple fractions $\frac{I}{(I-TR_\nu)^{k_\nu}}$, $= 1, \ldots, m$, the coefficients in the decomposition being endomorphisms commutative with the operation T and the generators. The solution of Eq. (6) is reduced to finding the results of the form $\frac{I}{(p-R_\nu)^{k_\nu}}g$ where $g \in G_1$.

Without change of generality we can formulate a similar theorem for a system of equations. Those results are valid particularly for equations with numerical coefficients [1]. The conditions

(A) $R_1, \ldots, R_m$ are generators,

(B) $x \neq 0 \supset T^{k_0}(I-TR_1)^{k_1} \ldots (I-TR_m)^{k_m}x \neq 0$

are equivalent.

4. Analysis of operators in linearly topological spaces

Let X be a linearly topological space and let $x_n \to x$ denote that the sequence x_n of elements of space X has the limit x. The linear space X can be extended to the linear space $X(T)$ as in sec. 3, taking in addition the similarities ax, where a is a number, as commutative endomorphisms. In the space of results $X(T)$ we introduce a new convergence by the formula

(10) $\operatorname{Lim} \dfrac{W_1(T)}{W_2(T)}x_n = \dfrac{W_1(T)}{W_2(T)}x$ if and only if there exists a polynomial $V(T)$ such that

$$\frac{W_1(T)V(T)}{W_2(T)}x_n \to \frac{W_1(T)V(T)}{W_2(T)}x,$$

assuming in addition that the following condition is satisfied:

(C) $x_n \to x, \quad y_n \to y, \quad Ax_n + By_n = 0 \supset Ax + By = 0 \quad$ for $\quad A, B \in \Pi[r]$.

With convergence (10) every operator $\dfrac{W_1(T)}{W_2(T)}$ is continuous. If $x_m = \sum_{k=0}^{n-1} \dfrac{U_k(p)}{U(p)}c_{k,m} + \dfrac{I}{U(p)}f_m$ and $c_{k,m} \to c_k$, $f_m \to f$ as $m \to \infty$, then we assume that $\operatorname{Lim} x_m = \sum_{k=0}^{n-1} \dfrac{U_k(p)}{U(p)}c_k + \dfrac{I}{U(p)}f$, owing to which Eq. (6) becomes posed analytically in a correct manner.

Condition (C) is satisfied in particular in locally convex linearly topological spaces if we take continuous endomorphisms as generators R_i [3]. We then have $\dfrac{I}{(p-R)^m}x = \sum_{n=0}^{\infty}\binom{n+m-1}{n}T^{n+m}R^n x$.

Space X is said to have a base if there exists a parametric function $\{\varepsilon_\sigma\}$ with values from space $X(T)$ such that every element x of space X can be uniquely written in the form of a sum $x = \sum_\sigma a_\sigma \varepsilon_\sigma$ (finite or infinite) or of an integral $x = \int_\Omega a_\sigma \varepsilon_\sigma d\sigma$, according to whether the parameter σ runs over the finite set of natural numbers, the infinite set of integers, or the subset of the m-dimensional space E_m. In particular

distributions $\delta(x-\sigma) = \frac{\partial}{\partial x}\begin{Bmatrix}1 \text{ for } x > \sigma\\ 0 \text{ for } x < \sigma\end{Bmatrix}$, $\sigma \in a, b$ and systems of functions $\varphi_\sigma(x)$, $\sigma = 1, 2, \ldots$ constituting sequences orthonormal and complete with the scalar product $(f(x), g(x)) = \int_a^b \overline{f(x)g(x)}\,dx$ prove to be bases in the spaces of continuous functions $f(x)$ in the interval a, b.

THEOREM 4. *If a space X has a base f_σ, and the space of those elements w for which $Sw = 0$ has a base e_σ, then it suffices if we solve the equation $Sx - Rx = f$, $\gamma x = x_0$, where $f \in X$, $Sx_0 = 0$, for the elements of the bases, obtaining Green functions $G_\sigma^0 = \frac{e_\sigma}{I - RT}$, $G_\sigma^1 = \frac{Tf_\sigma}{I - RT}$ of the first and the second kind. The complete solution is of the form $x = \sum_\sigma \xi_\sigma G_\sigma^0 + \sum_\sigma \varphi_\sigma G_\sigma^1$ (where $\xi_\sigma, \varphi_\sigma$ are numbers) with a denumerable set of parameters σ. In the opposite case the sign $\sum_\sigma$ of the sum should be replaced by the sign of integral $\int_\Omega \ldots d\sigma$.*

In particular, suppose that in the space of functions $u(t, x) \in X$ (with the ordinary multiplication $\alpha(t)\beta(x)$) three operations act: S_t, γ_t on the function $u(t, x)$ regarded as a function of the variable t, R_x — on $u(t, x)$ as a function of the variable x. Let R_x be a generator in space X.

Function $\varphi(x)$ is called the characteristic function of generator R_x and number λ the characteristic value if $R\{\varphi(x)\} = \lambda\{\varphi(x)\}$.

The method of separating the variables in space X can be formulated as follows:

THEOREM 5. *We consider the equation $S_t u - R_x u = \{f(t, x)\}$, $\gamma_t u = \{u_0(x)\}$ in space X. In this space let the characteristic functions φ_σ of generator R_x form a base in the subspace of functions $f(x)$ and let the similarities $\lambda_\sigma I$ corresponding to the characteristic values be generators in space X. Then the base φ_σ is a base for the whole space X, i. e. the solution of the equation $S_t u - R_x u = \{f(t, x)\}$, $\gamma u = \{u_0(x)\}$ is written in the form $u(t, x) = \sum_\sigma \alpha_\sigma(t)\varphi_\sigma(x)$, where the Fourrier coefficient $\alpha_\sigma(t)$ in the expansion of the solution with respect to the elements of the base is expressed by the formula*

$$\alpha_\sigma(t) = \frac{\{\xi_\sigma^0\} + T_t\{f_\sigma(t)\}}{I - \lambda(\sigma)\,T_t} \qquad \text{with} \qquad u_0(x) = \sum_\sigma \xi_\sigma^0 \varphi_\sigma(x), \quad f(x, t) = \sum_\sigma f_\sigma(t)\varphi_\sigma(x).$$

5. Connection of operation T with convolution

Suppose that in the linearly topological space discussed in sec. 4 a multiplication of elements $a*b$, called convolution, is defined, satisfying two conditions:

(D) space X forms under multiplication $*$ a commutative linearly topological ring,

(E) there exists such an element l in space X that for every element $f \in X$ we have the equality $Tf = l*f$.

Let us confine ourselves to the consideration of generators R satisfying the condition $RTf = Rl*f$ for every element $f \in X$. With the aid of those generators let us construct a set $\Pi^*[T]$ of polynomials $W(T) = aT^{k_0}(I - TR_1)^{k_1} \ldots (I - TR_m)^{k_m}$ where a is a number, and a space of results $X^*(T)$.

We can see that the convolution can be extended to the whole space of results $X^*(T)$ by the formula

$$(11) \qquad \frac{W_1}{W_2}f * \frac{W_3}{W_4}g = \frac{W_1 W_3}{W_2 W_4}(f*g) \quad \text{with} \quad W_1, W_2, W_3, W_4 \in \Pi^*[T],\ f, g \in X.$$

We obtain a linear ring, the operations of addition, multiplication by a number and convolution being continuous in the sense of convergence in a space of results.

Let the commutative operation U be termed a convolution operation if

$$(12) \qquad U(Tf) = Ul*f \quad \text{for every element} \quad f \in X.$$

THEOREM 6. *If U_1, U_2 are convolution operations and $h = U_1 f$, $k = U_2 g$, where $f, g, h, k \in X$, then $h*k = f*g$.*

Remark. In particular, the commutative operation U is a convolution operation if every element $f \in X$ can be represented in the form

$$f = \lim_{n\to\infty} (a_{n,0}l + a_{n,1}Tl + \ldots + a_{n,k_n}T^{k_n}l)\,,$$

where the coefficients $a_{n,i}$ are numbers.

6. Examples

A. Differential equations with constant coefficients. Consider the equation

$$a_n x^{(n)}(t) + a_{n-1}x^{(n-1)}(t) + \ldots + a_0 x(t) = f(t)\,,$$

where $x(t)$ is a function n times differentiable in a continuous manner in a fixed interval a, b (it may be an open interval and also an improper interval).

The space of continuous function should be taken as the space X and the operation of differentiating as operation S. The equation $y'(t) = x(t)$ will be uniquely solvable if we impose the boundary condition $\gamma\{y(t)\} = \{y(t_0)\}$, where $t_0 \in a, b$. Operation T has the form $T\{x(t)\} =$

$= \left\{\int_{t_0}^{t} x(\tau)\,d\tau\right\}$, $\gamma(S^k x) = \{x^{(k)}(t_0)\}$; for every complex number a the result $\frac{I}{p-a} f (f \in X)$ is unique. The theorems on convolution yield the theorems of Duhamel and Borel known from the theory of Laplace transform.

Modifying slightly operation S, we can assume that the function $f(t)$ is continuous by intervals and seek the solution under the boundary conditions defining the discontinuities of the function and the derivatives up to and including the $(n-1)$-th one.

B. Harmonic equations with constant coefficients. Consider in a simply connected domain D the equation $a_n \Delta_z^n z(x, y) + a_{n-1}\Delta^{n-1} z(x, y) + \ldots + a_0 z(x, y) = f(x, y)$, where $z(x, y)$ is a function possessing an n-th iteration of the Laplacian $\Delta z = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$ continuous in the closure of the domain D.

The space of functions $z(x, y)$ continuous in the domain D should be taken as the space X, the operation of the Laplacian as operation S. The equation $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = f(x, y)$ will be uniquely solvable if we impose the boundary condition $\gamma\{z(x, y)\} = \varphi(x, y)$, where $\varphi(x, y)$ is a harmonic function. It is, of course, sufficient to give the values $z|_\Gamma$ of the function $\varphi(x, y)$ on the boundary Γ of the domain D.

Since the equation $\Delta z = a \cdot z, z|_\Gamma = 0$ is uniquely solvable only for $a \leqslant 0$, the result $\frac{I}{p-a} f$ is not always unique. As an application we obtain the following theorem on harmonic equations:

The solution of equation $a_n \Delta^n z + a_{n-1}\Delta^{n-1} z + \ldots + a_0 z = f$ *with the boundary conditions* $\Delta^k z(x, y)|_\Gamma = \psi_k(x, y)$, $k = 0, 1, \ldots, n-1$ *is unique if* $\operatorname{sgn} a = -\operatorname{sgn} a_{n-1} = \ldots = (-1)^n \operatorname{sgn} a_0$.

It can be obtained by the Heaviside method by a superposition of the solution of the equations $\Delta z = az + \psi(x, y)$, $z|_\Gamma = 0$ for $a \leqslant 0$.

C. Consider the equation $x^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + x \frac{\partial u}{\partial x} = 0$ with the initial conditions $u(1, y) = \varphi(y)$, $u'_x(1, y) = \psi(y)$.

It follows from the theory discussed above that under the operations $Su = x \frac{\partial u}{\partial x}$, $Ru = \frac{\partial u}{\partial y}$, $\gamma u = \{u(1, y)\}$ we have $Tu = \left\{\int_1^x \frac{u(\sigma, y)}{\sigma} d\sigma\right\}$, $\frac{I}{I - TR}\{f(y)\} = \{f(\pm \ln x + y)\}$ and the solution of the equation can be written in the form

$$u(x, y) = \frac{\varphi(y + \ln x) + \varphi(y - \ln x)}{2} + \frac{1}{2} \int_{y-\ln x}^{y+\ln x} \frac{\psi(\sigma)}{\sigma} d\sigma.$$

D. Consider the space of functions $x(\lambda)$ of the parameter λ defined on the interval a, b with values from the field of Mikusiński operators, continuous in the sense of Mikusiński. Using the Mikusiński derivative we come to a space of results. The multiplication by operator k is a generator if k is a logarithm in the sense of Mikusiński [2].

E. Let X be a locally convex linearly topological space. Every constant continuous endomorphism is a generator in the space X.

This model corresponds to the theory of ordinary differential equations discussed by W. Słowikowski [3].

F. S. Bellert [1] obtains the Heaviside algorithm for equations with constant numerical coefficients in each of the models he considers. As the set of generators we should take here the set of similarities.

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MATHEMATICS

Some Remarks on the Marcinkiewicz-Orlicz Space. II

by

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Presented by W. ORLICZ on October 28, 1958

We denote by $\mathcal{O}$ a Marcinkiewicz-Orlicz space defined in the preceding note [1]. By a norm $\|\ \|$ of an element $o \in \mathcal{O}$ we mean a pseudonorm defined in [1], formula (2) — of an arbitrary representative of o. It is easily seen that the elements of $\mathcal{O}$ are sets composed of equivalent functions *). Therefore it seems quite natural to term likewise trigonometrical polynomial elements of $\mathcal{O}$ having trigonometrical polynomials as representatives.

Now, we denote by $\mathcal{B}$ the subspace of $\mathcal{O}$ composed of all trigonometrical polynomials and closed under the norm.

THEOREM 1. *The $\mathcal{B}$-space is non-separable Banach space.*

THEOREM 2. *If $o \in \mathcal{B}$, and if x is an arbitrary representative of o, then there exist*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,dt \quad \text{and} \quad \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|\,dt\,;$$

the values of these limits are the same for all representatives of o.

THEOREM 3. *If x is a representative of an arbitrary element o belonging to $\mathcal{B}$, there exists for every real number λ*

$$a(\lambda) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t) e^{-i\lambda t}\,dt$$

and the value of this limit is the same for all representatives of o.

The number $a(\lambda)$ is called the Fourier coefficient of o.

THEOREM 5. *Every element belonging to $\mathcal{B}$ has at most a denumerable set of Fourier coefficients, different from zero.*

*) Two functions are called equivalent if their difference belongs to Q (for definition of the latter see [1]).

Therefore, in the usual way, we may subordinate the Fourier series to every element belonging to $\mathcal{B}$.

To every element $o \in \mathcal{B}$, we may subordinate a Bochner-Fejér polynomial by the formula

$$\sigma_B(x) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t+\tau) K_B(\tau)\, d\tau\,,$$

wherein $K_B(\tau)$ denotes a Bochner-Fejér kernel, and x is an arbitrary representative of o, $\sigma_B(x)$ is a representative of the element $\sigma_B(o)$, belonging to $\mathcal{B}$.

THEOREM 6. *For every element o belonging to $\mathcal{B}$ we have*

$$\|\sigma_B(o)\| \leqslant \|o\|\,.$$

THEOREM 7. *For every real number $\varepsilon > 0$ and every $o \in \mathcal{B}$ there exists a Bochner-Fejér polynomial $\sigma_B(o)$, such that*

$$\|\sigma_B(o) - o\| \leqslant \varepsilon\,.$$

THEOREM 8. *If two elements o_1, o_2 from $\mathcal{B}$ have the same Fourier series, then $o_1 - o_2 \in Q$.*

If the function M fulfills condition Δ_2, for $|u| \geqslant 0$ (for definition of the latter, see [2], p. 36) we may characterize representatives of all elements belonging to $\mathcal{B}$ by their structural properties. These properties are analogous to the structural properties of almost-periodic functions in Besicovitch's sense. All the above given results were communicated at the Conference on Stochastic Processes and Almost-Periodic Functions held at Wrocław in September, 1955.

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MATHEMATICS

Linear Equations and Pure Subgroups

by

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Presented by K. KURATOWSKI on October 29, 1958

1. Linear equations; $\varkappa$-pure subgroups. By a system of linear equations of power $\mathfrak{m}$ over the group G *), we understand a system of the form

$$(U) \qquad \sum_{s \in T} a_{t,s} x_s = g_t \,,$$

where t runs over the set T, which is of power $\mathfrak{m}$, $a_{t,s}$ are integers, g_t are elements of G and, for t fixed, $a_{t,s} \neq 0$ holds only for a finite number of s in T.

Let G be a subgroup of the group H, and let $\varkappa$ be an ordinal. The subgroup G is called $\varkappa$-pure in H if every system of linear equations of power $\mathfrak{m} < \aleph_\varkappa$ with coefficients in G, which has a solution in H, has also a solution in G.

It is known (see Prüffer [6]), that the notion of a 0-pure subgroup coincides with the usual notion of a pure subgroup (i. e. $nG = nH \cap G$, for every n).

The generalization of the notion of pure subgroups is due to S. Gacsályi. It follows from the results contained in the paper [3] and [1] that:

> G is $\varkappa$-pure in H if and only if G is a direct summand of every subgroup H_0 with $G \subset H_0 \subset H$, in which G is of index $< \aleph_\varkappa$ **).

We shall call a system (U) of linear equations over the group G a Balcerzyk system of the groups G and H (where, as before, G is a subgroup of H), if with every subgroup H_0 of H, which contains G, a subsystem (U, H_0) of (U) may be associated in such a manner that the following conditions are satisfied:

(i) $(U, H) = (U)$.

(ii) (U, H_0) is finite if $\operatorname{ind}(G, H_0)$ is finite, or of the power $\operatorname{ind}(G, H_0)$.

*) All groups concerned in this paper are assumed to be additively written Abelian groups.

**) The index of G in H_0 is the order of H_0/G: $\operatorname{ind}(G, H_0) = |H_0/G|$.

(iii) (U, H_0) is solvable in G if and only if G is a direct summand of H_0.

(iv) for every subsystem (U_0) of (U) with cardinality $\mathfrak{m} \geqslant \aleph_0$, there exists a subgroup H_0 of H such that $G \subset H_0$, $\operatorname{ind}(G, H_0) = \mathfrak{m}$ and (U_0) is contained in (U, H_0).

For every pair of groups $G \subset H$ there exists a Balcerzyk system of linear equations. This follows from the results contained in [1].

It may at once be seen that, if G is a $\varkappa$-pure subgroup of H, but not a direct summand, then every subsystem of Balcerzyk's system of G and H with cardinality $< \aleph_\varkappa$ is solvable in G, whereas the whole system has no solution in G.

2. The groups $S_0^\varkappa$ and $\mathcal{S}_\lambda^\varkappa$. By S_λ^* we shall denote the group of ω_λ-sequences *) of integers.

This group is what is called a complete direct sum of $\aleph_\lambda$ copies of an infinite cyclic group.

Let $x = (x_\xi)$ belong to S_λ^*; by $\nu(x)$ we shall denote the support of x, i. e. the set of these ordinals $\xi < \omega_\lambda$ for which $x_\xi \neq 0$: $\nu(x) = \{\xi < \omega_\lambda; x_\xi \neq 0\}$.

For an ordinal $\varkappa$, we denote by $S_\lambda^\varkappa$ the group of all elements x in S_λ^* with support $\nu(x)$ of a power less than $\aleph_\varkappa$: $S_\lambda^\varkappa = \{x \in S_\lambda^*;\ |\nu(x)| < \aleph_\varkappa\}$.

We have, of course, $S_\lambda^{\lambda+1} = S_\lambda^*$. We put, moreover, $S_\lambda = S_\lambda^0$. S_λ is a free (Abelian) group with $\aleph_\lambda$ generators.

A cardinal $\mathfrak{m}$ is called of measure 0 if, in a set T of power $\mathfrak{m}$, there does not exist any σ-measure with values 0 and 1 only, defined for all subsets of T and vanishing for one-point sets. It is known (see Ulam [7]), that all cardinals smaller than the first strongly inaccessible aleph are of measure 0. It follows that $2^{\aleph_0} = \mathfrak{c}$, $2^{\mathfrak{c}} = \mathfrak{f}$; $2^{\mathfrak{f}}$ and so on, are of measure 0. In the sequel we shall need the following lemma (see Ehrenfeucht-Łoś [2]):

If $\aleph_\lambda$ is of measure 0, then every homomorphism mapping S_λ^* into the infinite cyclic group C_∞ and vanishing on S_λ (i. e. $h(x) = 0$ for x in S_λ), vanishes identically on the whole group S_λ^*.

It follows from this

LEMMA. *If $\aleph_\lambda$ is of measure* 0, *then no subgroup G of S_λ^* containing S_λ is a direct summand in S_λ^*.*

Proof: Let $S_\lambda^* = A + B$, $S_\lambda \subset A$ and let $x^0 \neq 0$ belong to B. It follows that, for a suitable $\xi_0 < \omega_\lambda$, we have $x_{\xi_0}^0 \neq 0$. Let us set for every $x = a + b$ with a in A and b in B: $h(x) = b_{\xi_0}$. The mapping h is, of course, a homomorphism of S_λ^* into C_∞ and, for x in A (then all the more for x in S_λ), we have $h(x) = 0$. For x^0, we have $h(x^0) = x_{\xi_0}^0 \neq 0$, therefore,

*) ω_λ is the first ordinal of the power $\aleph_\lambda$, a ω_λ-sequence is a sequence of the type ω_λ. In the group S_λ^* the addition will be performed on co-ordinates, i. e. $(x+y)_\xi = x_\xi + y_\xi$, for all $\xi < \omega_\lambda$.

h does not vanish on the whole group S_λ^*, which contradicts the lemma above.

Now we shall prove the following *):

(*) THEOREM 1. *If $\aleph_\varkappa$ is regular* **), *then $S_\lambda^\varkappa$ is $\varkappa$-pure in S_λ^*.*

Proof: Suppose that the system

$$\text{(I)} \qquad \sum_{s\in T} a_{t,s}x_s = p^{(t)} \in S_\lambda^\varkappa, \qquad t \in T,$$

where $|T| < \aleph_\varkappa$, has a solution $q^{(t)}$ in S_λ^*. Then, for every $\xi < \omega_\lambda$, we have

$$\text{(II)} \qquad \sum_{s\in T} a_{t,s}q_\xi^{(s)} = p_\xi^{(t)}.$$

Let Z be the union of all $v(p^{(t)})$. We have $|v(p^{(t)})| < \aleph_\varkappa$ and — for $\aleph_\varkappa$ — regular — $|Z| \leqslant \sum_{t\in T} |v(p^{(t)})| < \aleph_\varkappa$.

Let us set $r_\xi^{(t)} = q_\xi^{(t)}$ for ξ in Z and $r_\xi^{(t)} = 0$ for ξ which does not belong to Z. Clearly, $v(r^{(t)}) \subset Z$, therefore, $r^{(t)}$ are in $S_\lambda^\varkappa$. But $r^{(t)}$ form a solution of (I), since

$$\sum_{s\in T} a_{t,s}r_\xi^{(s)} = p_\xi^{(t)}$$

follows from (II) if ξ belongs to Z, and $p_\xi^{(t)} = 0$ if ξ does not belong to Z.

We have found for the system (I) a solution in $S_\lambda^\varkappa$ and, therefore, our theorem is proved.

We shall call a group G C_∞-regular, or shortly regular, if, for every $g \neq 0$ in G, there exists a homomorphism h of G into C_∞, such that $h(g) \neq 0$. Those groups are regular which are isomorph to the subgroups of S_λ^*. A group which has no homomorphisms into C_∞ (except the trivial one) will be called completely irregular. It follows from the lemma given in section **1** that, for $\aleph_\lambda$ of measure 0, if $S_\lambda \subset G$, then S_λ^*/G is a completely irregular group.

By $\mathcal{S}_\lambda^\varkappa$ we shall denote the factor group $S_\lambda^*/S_\lambda^\varkappa$. For $\aleph_\lambda$ of measure 0 it is, of course, a completely irregular group.

(*) THEOREM 2. *If $\aleph_\varkappa$ is a regular aleph, then every subgroup $\mathcal{G}$ of $\mathcal{S}_\lambda^\varkappa$ of order less than $\aleph_\varkappa$ is a regular group.*

Proof: $\mathcal{G} = H/S_\lambda^\varkappa$ and $\operatorname{ind}(S_\lambda^\varkappa, H) < \aleph_\varkappa$, then $S_\lambda^\varkappa$ is a direct summand of H: $H = S_\lambda^\varkappa + G$. The group $\mathcal{G}$ is isomorphic to G and it is a subgroup of S_λ^*, therefore $\mathcal{G}$ is regular.

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*) The theorems marked with one asterisk were presented at the II Congress of Russian Mathematicians in Moscow, June-July 1956, and are contained in my paper [5]. The theorems marked with two asterisks were presented at the International Congress of Mathematicians, Edinburgh, 1958.

**) $\aleph_\varkappa$ is called regular, if $|T| < \aleph_\varkappa$ and $\mathfrak{m}_t < \aleph_\varkappa$ implies $\sum_{t\in T} \mathfrak{m}_t < \aleph_\varkappa$. If $\varkappa = \mu + 1$, then $\aleph_\varkappa$ is regular.

COROLLARY. *If $\aleph_\lambda$ is of measure 0 and $\aleph_\varkappa$ is regular, then the order of every direct summand of $\mathcal{S}^\varkappa_\lambda$ is not less than $\aleph_\varkappa$; i. e. if $\mathcal{S}^\varkappa_\lambda = \mathcal{A}+\mathcal{B}$ and $\mathcal{A} \neq (0) \neq \mathcal{B}$, then $|\mathcal{A}| \geqslant \aleph_\varkappa$ and $|\mathcal{B}| \geqslant \aleph_\varkappa$.*

Proof: Suppose $\mathcal{S}^\varkappa_\lambda = \mathcal{A}+\mathcal{B}$ and $|\mathcal{A}| < \aleph_\varkappa$, then $\mathcal{A}$ is a regular group and therefore there exists a homomorphism h of $\mathcal{A}$ onto C_∞. Let us set for $g = a+b$, where a is in $\mathcal{A}$ and b is in $\mathcal{B}$, $f(g) = h(a)$. The mapping f is of course a homomorphism of $\mathcal{S}^\varkappa_\lambda$ onto C_∞, which is impossible, since for the group $\mathcal{S}^\varkappa_\lambda$ is completely irregular.

3. $\varkappa$-pure cyclic groups. From Theorem 1 follows the existence of $\varkappa$-pure subgroups which are direct summands, for all such $\varkappa$ that $\aleph_\varkappa$ is regular and of measure 0. These are the groups $S^\varkappa_\lambda$. But from this example we do not know if, for arbitrary large $\varkappa$, there exist or not $\varkappa$-pure subgroups of limited order, f. i. of order $\aleph_0$.

In this section we shall prove that, for every $\varkappa$ such that $\aleph_\varkappa$ is regular and of measure 0, there exists a $\varkappa$-pure cyclic group, which is not a direct summand.

By the diagonal $\mathcal{D}$ in the group $\mathcal{S}^\varkappa_\lambda$ we mean the group consisting of elements $c+S^\varkappa_\lambda$, where c is a constant sequence in S^*_λ. The group $\mathcal{D}$ is obviously a cyclic infinite one; if $x+S^\varkappa_\lambda$ is a generator of $\mathcal{D}$, then the set of $\xi < \omega_\lambda$ with $x_\lambda \neq \pm 1$ is of a cardinality less than $\aleph_\varkappa$.

(**) THEOREM 3. *If $\aleph_\varkappa$ is regular, then the diagonal subgroup $\mathcal{D}$ is $\varkappa$-pure in $\mathcal{S}^\varkappa_\lambda$.*

Proof: Using the second definition of $\varkappa$-pure subgroups we shall demonstrate that $\mathcal{D}$ is a direct summand of every group $\mathcal{G}$ with $\mathcal{D} \subset \mathcal{G} \subset \mathcal{S}^\varkappa_\lambda$ and of order less than $\aleph_\varkappa$ ($\mathcal{D}$ is denumerable, therefore $\operatorname{ind}(\mathcal{D}, \mathcal{G}) = |\mathcal{G}|$ for every $\mathcal{G} \neq \mathcal{D}$).

Let $\mathcal{G} \subset \mathcal{S}^\varkappa_\lambda$ be of an order less than $\aleph_\varkappa$ and let $\mathcal{G}$ contain $\mathcal{D}$. Group $\mathcal{G}$ is of the form $H/S^\varkappa_\lambda$ and since $\operatorname{ind}(S^\varkappa_\lambda, H) = |\mathcal{G}| < \aleph_\varkappa$, hence $H = S^\varkappa_\lambda + G$. Let $g^{(0)}$ be such an element of G, that $g^{(0)}+S^\varkappa_\lambda$ is the generator of $\mathcal{D}$. Then there exists an isomorphism φ which maps G onto $\mathcal{G}$ in such a manner that $\varphi(g^{(0)}) = g^{(0)}+S^\varkappa_\lambda$. In consequence we have $\varphi(\{g^{(0)}\}) = \mathcal{D}$ (by $\{g^{(0)}\}$ is denoted the cyclic group of $g^{(0)}$). The sequence $g^{(0)} = (g^{(0)}_\xi)_{\xi<\omega_\lambda}$ is $\neq \pm 1$ only in a number of places less than $\aleph_\varkappa$, therefore, $g^{(0)}_{\xi_0} = 1$ for some ξ_0, and it follows that $\{g^{(0)}\}$ is a pure (0-pure) subgroup of S^*_λ. It is known that a pure cyclic subgroup of S^*_λ is a direct summand of S^*_λ, therefore, also of G. But if $\{g^{(0)}\}$ is a direct summand of G, then $\varphi(\{g^{(0)}\}) = \mathcal{D}$ is a direct summand of $\mathcal{G}$. The theorem is proved.

We see that, if $\aleph_\lambda$ is of measure 0, and $\aleph_\varkappa$ is regular, then $\mathcal{D}$ is a cyclic $\varkappa$-pure subgroup of $\mathcal{S}^\varkappa_\lambda$ which is not a direct summand (since $\mathcal{S}^\varkappa_\lambda$ is completely irregular).

By a similar reasoning as above, a somewhat more general theorem may be proved:

THEOREM 3'. *If $\aleph_\varkappa$ is regular, then every pure (0-pure) cyclic subgroup of $\mathcal{S}_\lambda^\varkappa$ is $\varkappa$-pure.*

4. Applications to the products of metric spaces. Let $\aleph_\varkappa$ be regular, $\aleph_\lambda$ of measure 0, and let (U) be Balcerzyk's system of the group $\mathcal{D}$ and $\mathcal{S}_\lambda^\varkappa$. The group $\mathcal{D}$ is a cyclic one, therefore (U) may be regarded as a system of linear equations with integral coefficients.

Since $\mathcal{D}$ is a $\varkappa$-pure subgroup of $\mathcal{S}_\lambda^\varkappa$ but not a direct summand, hence it follows from the properties of Balcerzyk's systems that

(i) (U) has no integral solution,

(ii) every subsystem (U_0) of (U) with cardinality less than $\aleph_\varkappa$ has an integral solution.

The rank of $\mathcal{S}_\lambda^\varkappa$ is $2^{\aleph_\lambda}$ and so is the cardinality of the system (U). Let $\aleph_\mu = 2^{\aleph_\lambda}$. We can assume that the set of indices T in (U) is the set of ordinals less than ω_μ: $T = \{\xi; \xi < \omega_\mu\}$. Then the integral solutions of every subsystems (U_0) of (U) may be regarded as subsets of S_μ^*.

A topological space $\mathfrak{X}$ will be called $\varkappa$-compact, if every family M of closed subsets of $\mathfrak{X}$ with void intersection: $\bigcap_{F \in M} F = 0$, contains a subfamily M_0 of cardinality less than $\aleph_\varkappa$ with void intersection.

0-compactness means ordinary compactness (bicompactness),

1-compactness is equivalent to Lindelöff's condition.

(**) THEOREM 4. *If $\aleph_\varkappa$ is a regular aleph of measure 0, then the product space of $\aleph_\mu = 2^{\aleph_\varkappa}$ not 0-compact metric spaces is not $\varkappa$-compact.*

Proof: Let $\mathfrak{X}_\xi$, $\xi < \omega_\mu$ are not 0-compact metric spaces. Every $\mathfrak{X}_\xi$ contains a denumerable discrete set F_ξ closed in $\mathfrak{X}_\xi$. Suppose the product space $P_{\xi<\omega_\mu} \mathfrak{X}_\xi$ to be $\varkappa$-compact, then the subspace $P_{\xi<\omega_\mu} F_\xi$ is also $\varkappa$-compact.

Consider S_μ^*; it is a product of $\aleph_\mu$ denumerable sets, namely of $\aleph_\mu$ copies of the set of integers, which are the axes of S_μ^*. Every axis of S_μ^* may be considered as a discrete topological space, and then the product topology may be introduced in S_μ^*. In this topology S_μ^* is homeomorphic to $P_{\xi<\omega_\mu} F_\xi$ and, therefore, it is enough to show that S_μ^* is not $\varkappa$-compact.

Let (U) be Balcerzyk's system of $\mathcal{D}$ and $\mathcal{S}_\lambda^\varkappa$. For a subsystem (U_0) of (U) we denote by $S(U_0)$ the integral solutions of (U_0), therefore, $S(U_0)$ is a subset of S_μ^* and it is easy to see that $S(U_0)$ is a closed set in S_μ^*.

Let $\{(U_m)\}_{m \in M}$ be the family of all subsystems of (U) with cardinality less than $\aleph_\varkappa$. Consider the family $\{S(U_m)\}_{m \in M}$. It is a family of closed sets in S_μ^* and the intersection of this family is void, for

$$\bigcap_{m \in M} S(U_m) = S(U) = 0.$$

Let M_0 be a subset of M with cardinality less than $\aleph_\varkappa$ and let (U_0) be the join of the systems (U_m), $m \in M_0$. The system (U_0) is of a power less than $\aleph_\varkappa$ and, therefore, we have

$$\bigcap_{m \in M_0} S(U_m) = S(U_0) \neq 0 .$$

We have shown the space S_μ^* to be not $\varkappa$-compact and, therefore, our theorem is proved.

Theorem 4 is connected with some results of E. Hewitt on Q-spaces [4].

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MATHEMATICS

Generalized Limits in Algebraically Compact Groups

by

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By a generalized λ-limit in an Abelian group G we understand a functional which assigns to every sequence $\{x_\xi\}_{\xi<\omega_\lambda}$ of the type ω_λ *) of elements in G an element $\operatorname{Lim}_{\xi<\omega_\lambda} x_\xi$ in G, so that

(I) $\operatorname{Lim}_{\xi<\omega_\lambda}(x_\xi+y_\xi)=\operatorname{Lim}_{\xi<\omega_\lambda} x_\xi+\operatorname{Lim}_{\xi<\omega_\lambda} y_\xi$.

(II) If $x_\xi=g$, for $\xi<\omega_\lambda$, then $\operatorname{Lim}_{\xi<\omega_\lambda} x_\xi=g$.

(III) If there exists an ordinal $\xi_0<\omega_\lambda$ such that $x_\xi=y_\xi$ for $\xi\geqslant\xi_0$, then $\operatorname{Lim}_{\xi<\omega_\lambda} x_\xi=\operatorname{Lim}_{\xi<\omega_\lambda} y_\xi$.

The purpose of this paper is to show that λ-limits exist for all λ only in algebraically compact groups.

The concept of an algebraically compact Abelian group was introduced by I. Kaplansky [2]. As shown in papers [1] and [3], the following definition is adequate for this concept:

G is algebraically compact if and only if G is a direct summand of every Abelian group H containing G as a pure subgroup.

Using this definition, we shall demonstrate the following

THEOREM. *In an Abelian group G there exists a generalized λ-limit for every λ if and only if G is algebraically compact.*

Proof: Let G be an algebraically compact group and S_λ^* the group of all sequences $x=\{x_\xi\}_{\xi<\omega_\lambda}$ with x_ξ in G. The addition in S_λ^* is defined as usually, i. e. as addition by terms. We denote by S_λ^0 the subgroup of S_λ^* formed by all the sequences for which there exists an ordinal $\xi_0<\omega_\lambda$ such that $x_\xi=0$ for $\xi\geqslant\xi_0$, and by D_λ the diagonal subgroup of S_λ^*, i. e. the subgroup of all constant sequences.

*) ω_λ is the first ordinal of power $\aleph_\lambda$.

The groups S_λ^0 and D_λ are disjoint and therefore the group $\boldsymbol{D}_\lambda = (D_\lambda + S_\lambda^0)/S_\lambda^0$ is isomorphic to D_λ and hence to G. It is easy to see that $\boldsymbol{D}_\lambda$ is a pure subgroup of S_λ^*/S_λ^0, and since D_λ is algebraically compact we have

$$S_\lambda^*/S_\lambda^0 = \boldsymbol{A} + \boldsymbol{D}_\lambda = A/S_\lambda^0 + (D_\lambda + S_\lambda^0)/S_\lambda^0 .$$

This shows that, for every x in S_λ^*, we have $x + S_\lambda^0 = (a + S_\lambda^0) + (d + S_\lambda^0)$ with a in A and d in D_λ; moreover, that in this decomposition d is uniquely determined by x. Therefore, we can put $\operatorname{Lim}_{\xi<\omega_\lambda} x_\xi = d_0$ for $x = \{x_\xi\}_{\xi<\omega_\lambda}$, where all terms of d equal d_0, and so we obtain the required limit.

To prove the converse, we proceed by induction on the power of the set of generators of H/G.

Suppose that, for every λ, there exists a generalized λ-limit in G, and that G is a pure subgroup of H. If H/G is finitely generated, then, as known, $H = A + G$ follows from the purity of G in H. If H/G is generated by a set of the power $\overline{\omega_\lambda}$, then there exists a sequence $G = H_0 \subset H_1 \subset \ldots \subset H_\xi \subset \ldots$, $\xi < \omega_\lambda$, of subgroups of H such that $\bigcup_{\xi<\omega_\lambda} H_\xi = H$ and every H_ξ/G is generated by a set of a power $< \overline{\omega_\lambda}$.

As our inductive assumption we can put $H_\xi = A_\xi + G$ for $\xi < \omega_\lambda$. Let us choose from every coset of H/G an element h_t. If $h_t + G$ belongs to the group H_ξ/G, then $h_t = g_t^{(\xi)} + a_t^{(\xi)}$ with some $g_t^{(\xi)}$ in G and $a_t^{(\xi)}$ in A. If $h_t + G$ does not belong to H_ξ/G, we put $g_t^{(\xi)} = 0$.

In this way we have defined, for every class $h_t + G$ in H/G, a sequence $g_t^{(\xi)}$. Let $\operatorname{Lim}_{\xi<\omega_\lambda} g_t^{(\xi)} = g_t$ and $h_t - g_t = a_t$. We denote by A the set of all these a_t.

From each class in H/G one and only one a_t is selected. Therefore, in order to prove $H = A + G$, it is enough to show that A is a group, or — equivalently — that if $(h_{t_1} + G) + (h_{t_2} + G) = (h_{t_3} + G)$, then $a_{t_1} + a_{t_2} = a_{t_3}$.

If $(h_{t_1} + G) + (h_{t_2} + G) = (h_{t_3} + G)$, then $h_{t_1} + h_{t_2} - h_{t_3} = g \in G$. For sufficiently large ξ we have $h_{t_i} = g_{t_i}^{(\xi)} + a_{t_i}^{(\xi)}$ $(i = 1, 2, 3)$ and (the $a_{t_i}^{(\xi)}$ are in A_ξ) $a_{t_1}^{(\xi)} + a_{t_2}^{(\xi)} - a_{t_3}^{(\xi)} = 0$. It follows that $g_{t_1}^{(\xi)} + g_{t_2}^{(\xi)} - g_{t_3}^{(\xi)} = g$. On the other hand, $a_{t_1} + a_{t_2} - a_{t_3} = h_{t_1} + h_{t_2} - h_{t_3} - (\operatorname{Lim}_{\xi<\omega_\lambda} g_{t_1}^{(\xi)} + \operatorname{Lim}_{\xi<\omega_\lambda} g_{t_2}^{(\xi)} - \operatorname{Lim}_{\xi<\omega_\lambda} g_{t_3}^{(\xi)}) = g - \operatorname{Lim}_{\xi<\omega_\lambda} (g_{t_1}^{(\xi)} + g_{t_2}^{(\xi)} - g_{t_3}^{(\xi)})$. The latter limit is $= g$, since the terms with sufficiently large ξ are $= g$. It follows $a_{t_1} + a_{t_2} - a_{t_3} = 0$ and $a_{t_1} + a_{t_2} = a_{t_3}$.

Evidently, our theorem can also be expressed as follows:

The group G is algebraically compact if and only if for every λ we have: $S_\lambda^* = A + D_\lambda$ and $S_\lambda^0 \subset A$.

It may be remarked that, for every G, the group S_λ^* can be directly decomposed into D_λ and a suitable group A, but the condition $S_\lambda^0 \subset A$ can be realized for algebraically compact groups only.

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MATHEMATICS

Construction of Directly Indecomposable Abelian Groups of Power Higher than That of the Continuum. II

by

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In a previous note [3] under the same title I have constructed directly indecomposable Abelian groups of power 2^c.

Other examples of directly indecomposable Abelian groups of power 2^c have been found by L. Fuchs [1], J. de Groot (not yet published) and A. Hulanicki [2].

The purpose of this note is to prove that for $\aleph_1$ distinct cardinals $\mathfrak{n}$ there exist directly indecomposable Abelian groups of power $\mathfrak{n}$. More precisely: let $\mathfrak{n}_0 < \ldots < \mathfrak{n}_\alpha < \mathfrak{n}_{\alpha+1} < \ldots$ $(\alpha < \omega_1)$ be the sequence of cardinals such that $\mathfrak{n}_0 = 1$, $\mathfrak{n}_{\alpha+1} = 2^{\mathfrak{n}_\alpha}$ and for a limit ordinal α let the cardinal $\mathfrak{n}_\alpha$ be the sum of all previous cardinals.

THEOREM. *For every cardinal $\mathfrak{n}_\alpha$ with $\alpha < \omega_1$ there exist directly indecomposable Abelian groups G_α of the rank $\mathfrak{n}_\alpha$.*

1. Notations and lemmas. By a group we shall mean an additively written Abelian group. We denote by R_p (p is a prime) the additive group of all rationals k/p^n with integer k, by $R_{p,q}$ (p, q distinct primes) the additive group of all rationals $k/p^n q^m$ with integer k. $\sum_{t \in T} A_t$ will denote the discrete direct sum of the groups $A_t (t \in T)$ and $\sum_{t \in T}^{*} A_t$ the complete direct sum of the same groups. The power of the set T will be denoted by $|T|$.

Now we summarize some lemmas without proofs:

(1) *To every torsion-free group H there exists a group $\widetilde{H}$ containing H such that $H = 2\widetilde{H}$.*

(2) *If H is a fully characteristic subgroup of $A+B$, then $H = A_1 + B_1$, where $A_1 \subset A$ and $B_1 \subset B$.*

(3) *If H is a fully characteristic and indecomposable subgroup of $A+B$, then H must be contained in one of the direct summands A and B.*

(4) *(Theorem of J. Łoś). If the groups $A_t (t \in T)$ are torsion-free and the power $|T|$ is smaller than the first aleph inaccessible in the strict sense, then every homomorphism of $\sum_{t\in T}^{*} A_t$ into the group $R_{p,q}$ which maps $\sum_{t\in T} A_t$ onto $\{0\}$ is zero* *).

2. Construction of the groups G_α. Let P and Q be two disjoint infinite sets of odd primes both arranged in a sequence of type $\alpha < \omega_1$.

$$P = (p_0, p_1, \ldots, p_\mu, \ldots)_{\mu<\alpha} \qquad Q = (q_0, q_1, \ldots, q_\mu, \ldots)_{\mu<\alpha}.$$

Let $(T_\mu)_{\mu<\alpha}$ be a family of disjoint sets such that $|T_\mu| = \mathfrak{n}_\mu$.

For every ordinal $\mu < \alpha$ we put

$$U_\mu^* = \sum_{t\in T_\mu}^{*} R^{(t)}_{p_\mu q_\mu}, \qquad \text{where} \qquad R^{(t)}_{p_\mu q_\mu} = R_{p_\mu q_\mu} \qquad (t \in T_\mu).$$

From every group $R^{(t)}_{p_\mu q_\mu}$ we choose the subgroup $R^{(t)}_{p_\mu} = R_{p_\mu}$ and an element $u_\mu^{(t)} \in R^{(t)}_{p_\mu q_\mu}$ such that

$$u_\mu^{(t)} \overline{\in} 2R^{(t)}_{p_\mu q_\mu} \qquad (t \in T_\mu).$$

Let V_μ be the subgroup generated by the subgroups $U_\mu = \sum_{t\in T_\mu} R^{(t)}_{p_\mu q_\mu}$ and $S_\mu^* = \sum_{t\in T_\mu}^{*} R^{(t)}_{p_\mu}$ of U_μ^* and let

$$W_\mu = \{2V_\mu, (u_\mu^{(t)})_{t\in T_\mu}\} = \{U_\mu, 2S_\mu^*\}.$$

It is easy to see that

(5) if $\sum_{i=1}^{n} k_i u_\mu^{(t_i)} \in 2V_\mu$, then 2 divides each k_i $(i = 1, 2, \ldots, n)$.

Now we construct an ascending sequence of groups H_μ $(\mu \leqslant \alpha)$ in the following way:

(i) $H_0 = R_{p_0 q_0}$,

(ii) $H_{\mu+1}$ is the subgroup of $\widetilde{H}_\mu + W_\mu$, where $\widetilde{H}_\mu \supset H_\mu$ and $2\widetilde{H}_\mu = H_\mu$ such that $H_{\mu+1} = \{H_\mu + 2V_\mu, (a_\mu^{(t)} + u_\mu^{(t)})_{t\in T_\mu}\}$ and $a_\mu^{(t)}$ are chosen from $\widetilde{H}_\mu$ so that the cosets $a_\mu^{(t)} + H_\mu$ form a basis of $\widetilde{H}_\mu / H_\mu$.

(iii) for a limit ordinal μ we put $H_\mu = \bigcup_{\nu<\mu} H_\nu$.

The group G_α we define as the sum of all H_μ $(\mu \leqslant \alpha)$

(6) $G_\alpha = \bigcup_{\mu\leqslant\alpha} H_\mu$.

Since each H_μ $(\mu \leqslant \alpha)$ is the group of rank $\mathfrak{n}_\mu$, hence

(7) G_α is the group of rank $\mathfrak{n}_\alpha$.

3. Proof of the theorem. It is easy to see that

(8) if an element $g \in H_\mu$ $(\mu \leqslant \alpha)$ is divisible by each power of a prime p_ν, where $\nu < \mu$, then $g \in H_\nu$.

*) For the proof of this theorem of J. Łoś see [4].

From this and from (ii) it follows that

(9) if H_ν is a fully characteristic subgroup of H_μ $(\nu < \mu)$, then $H_{\nu+1}$ is again a fully characteristic subgroup of H_μ.

Since by (8) H_0 is a fully characteristic subgroup of every group H_μ $(\mu \leqslant \alpha)$, then (9) implies

(10) if $\nu < \mu \leqslant \alpha$, then H_ν is a fully characteristic subgroup of H_μ.

Observe further that from (5) and (ii) it follows that

(11) no element of the form $\sum_{i=1}^{n} k_i u_\mu^{(t_i)}$, where $(k_1, \ldots, k_n) = 1$, belongs to the subgroup $H_{\mu+1} \subset \widetilde{H}_\mu + W_\mu$.

We prove now by induction that

(12) every subgroup H_μ $(\mu \leqslant \alpha)$ of G_α is directly indecomposable.

Evidently, H_0 is a directly indecomposable group. We suppose that H_μ is directly indecomposable and prove that then $H_{\mu+1}$ is again directly indecomposable.

From (ii) we have

$$H_{\mu+1} = \{H_\mu + 2V_\mu,\ (a_\mu^{(t)} + u_\mu^{(t)})_{t \in T_\mu}\} \subset \widetilde{H}_\mu + W_\mu .$$

Suppose that

(13) $H_{\mu+1} = A + B$ and $A \neq \{0\} \neq B$.

Since by (10) H_μ is a fully characteristic directly indecomposable subgroup of $H_{\mu+1}$, then by (3) H_μ must be contained in one of the direct summands of $H_{\mu+1}$ say $H_\mu \subset A$.

Let $\bar{A}$, $\bar{B}$ be the minimal pure subgroups of $\widetilde{H}_\mu + W_\mu$ containing A and B, respectively. Then, since $H_\mu \subset A$, we obtain

$$\bar{A} + \bar{B} = \widetilde{H}_\mu + W_\mu \quad \text{and} \quad \widetilde{H}_\mu \subset \bar{A} .$$

Observe that the subgroup $2U_\mu \subset 2V_\mu$ is fully characteristic in $H_{\mu+1}$ since an element $g \in H_{\mu+1}$ belongs to $2U_\mu$ if and only if g is divisible by each power of the primes p_μ and q_μ.

By (2) and (13) we have

$2U_\mu = A_1 + B_1$, where $A_1 \subset A$, $B_1 \subset B$ and $U_\mu = \bar{A}_1 + \bar{B}_1$, where $\bar{A}_1 \subset A$, $\bar{B}_1 \subset B$ and $2U_\mu = 2\bar{A}_1 + 2\bar{B}_1$.

If $B_1 \neq \{0\}$, then there exists an element $u = \sum_{i=1}^{n} k_i u_\mu^{(t_i)}$, where $(k_1, \ldots k_n) = 1$ which belongs to the $\bar{B}_1$. Consider $g = \sum_{i=1}^{n} k_i (a_\mu^{(t_i)} + u_\mu^{(t_i)}) \in H_{\mu+1} = A + B$. Hence, $g = a + b$, where $a \in A$, $b \in B$. Since $\widetilde{H}_\mu \subset \bar{A}$, the element g decomposes in the group $\bar{A} + \bar{B} = \widetilde{H}_\mu + W_\mu$ into the sum $g = \sum_{i=1}^{n} k_i a_\mu^{(t_i)} + \sum_{i=1}^{n} k_i u_\mu^{(t_i)}$, where $\sum_{i=1}^{n} k_i a_\mu^{(t_i)} \in \bar{A}$ and $\sum_{i=1}^{n} k_i u_\mu^{(t_i)} \in \bar{B}$. From

here we obtain that $b = \sum_{i=1}^{n} k_i u_\mu^{(t_i)} \in H_{\mu+1}$ what is impossible in view of (11). We have proved that $2U_\mu \subset A$. Further we have $2S_\mu^* \subset A$. In a contrary case, the superposition of the projection of $2S_\mu^*$ into B and the projection of B into W_μ is a non-trivial homomorphism of $2S_\mu^*$ into $\sum_{t \in T_\mu}^{n*} R_{p_{\mu^\nu} q_\mu}^{(t)}$. This homomorphism vanishes on $2S_\mu \subset 2U_\mu$ which is impossible in view of the theorem of J. Łoś (4).

Finally, we have $\{H_\mu, 2U_\mu, 2S_\mu^*\} \subset A$. Since the minimal pure subgroup of $H_{\mu+1}$ containing $\{H_\mu, 2U_\mu, 2S_\mu^*\}$ is equal $H_{\mu+1}$, we obtain that $H_{\mu+1} \subset A$ which contradicts (13).

It remains to prove that if μ is a limit ordinal and every group H_ν $(\nu < \mu)$ is directly indecomposable, then H_μ is directly indecomposable also. This follows at once from the fact that every H_ν by (10) is a fully characteristic subgroup of H_μ and from (3).

From (7), (10), (12) and Lemma (3) we obtain that G_α is a directly indecomposable group of the rank $\aleph_\alpha$.

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MATHEMATICS

Comment on the Imbedding of Polyhedra in Euclidean Spaces

by

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Introduction

According to a recent result of Wu Wen-Tsün ([6] Theorems 4 and 4') a finite polyhedron X of dimension $n \neq 2$ can be imbedded in the $2n$-dimensional Euclidean space R^{2n} if and only if a certain cohomology class of the deleted symmetric square of X vanishes. A similar result has also been obtained by A. Shapiro ([5] Theorem 7.2).

Our first object in this note is to draw attention to an elementary version of Wu Wen-Tsün's imbedding theorem. The cohomological condition will be consecutively replaced by an inequality bearing on the category (in the sense of [2]) of a certain covering space associated with X, by a property closely related to the Lusternik-Schnirelmann-Borsuk theorem and, finally, by a statement similar to the Borsuk-Ulam theorem. We next show how Wu Wen-Tsün's result may be applied to obtain the answer to the following question raised by E. Marczewski [4]: Find for each natural n the least number $m = \varphi(n)$ such that each separable (eventually compact) space of dimension not greater than n be quasi-homeomorphic to a subset of the m-dimensional Euclidean space.

Elementary properties and imbedding

Let $f\colon \widetilde{E} \to E$ be a continuous map. A subset $A \subset E$ is called *evenly covered* by the pair $(\widetilde{E}, f)$ if $f^{-1}(A)$ is not empty and every component of $f^{-1}(A)$ is topologically mapped by f onto A. A *covering space* of E is a pair $(\widetilde{E}, f)$ consisting of a connected locally connected space $\widetilde{E}$ and a continuous map $f\colon \widetilde{E} \to E$ such that every point of E lies in an open set which is

evenly covered by $(\widetilde{E}, f)$. The *category* $\operatorname{cat}(\widetilde{E}, f)$ of a covering space $(\widetilde{E}, f)$ of E is the least integer $k \geqslant 1$ such that E may be covered by k open sets all of whose components are evenly covered by $(\widetilde{E}, f)$; if no such integer exists, we put $\operatorname{cat}(\widetilde{E}, f) = \infty$ [2].

Let X be an arbitrary space, $X \times X$ its Cartesian square and $\Delta = \{(x, x) | x \in X\}$ the diagonal of $X \times X$. The *deleted Cartesian square* $\widetilde{X}^*$ of X is the subspace $X \times X - \Delta$ of $X \times X$; its points are the ordered pairs (x_1, x_2) satisfying $x_1 \neq x_2$. The map $t\colon X \times X \to X \times X$ defined by $t(x_1, x_2) = (x_2, x_1)$ is a homeomorphism of period 2 which will be referred to as the *antipodal map*; obviously $t(\widetilde{X}^*) = \widetilde{X}^*$ and $t|\widetilde{X}^*$ will still be denoted by t. The *deleted symmetric square* X^* of X is obtained from $\widetilde{X}^*$ by identifying every point (x_1, x_2) with its antipode (x_2, x_1); the points of X^* are the unordered pairs $\{x_1, x_2\}$ satisfying $x_1 \neq x_2$ and the natural projection $p\colon \widetilde{X}^* \to X^*$ is defined by $p(x_1, x_2) = \{x_1, x_2\}$ [6]. If X is a connected polyhedron which is neither a point nor a segment, then $(\widetilde{X}^*, p)$ is a covering space of X^* in the above sense; if X is a segment or a point, we shall still consider $(\widetilde{X}^*, p)$ as a covering space of X^* with category 1 or 0, respectively. There is no loss of generality in assuming below that the polyhedron X is connected.

THEOREM 1. *Let X be a finite connected polyhedron of dimension $n \neq 2$. Then the following statements are all equivalent*:

(i) *X can be imbedded in R^{2n}*;

(ii) $\operatorname{cat}(\widetilde{X}^*, p) \leqslant 2n$;

(iii) *$X \times X$ may be covered by $2n+1$ closed subsets none of which contains a pair of distinct antipodal points*;

(iv) *There exists a map $g\colon X \times X \to R^{2n}$ such that $x_1 \neq x_2$ implies $g(x_1, x_2) \neq g(x_2, x_1)$.*

Proof. For any imbedding $h\colon X \to R^{2n}$ let

$$\tilde{f}(x_1, x_2) = \frac{h(x_2) - h(x_1)}{\|h(x_2) - h(x_1)\|} \in S^{2n-1} \quad \text{if} \quad x_1 \neq x_2 .$$

Then the first square of the diagram

(1)
$$\begin{array}{ccccc} \widetilde{X}^* & \xrightarrow{t} & \widetilde{X}^* & \xrightarrow{p} & X^* \\ \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f} \\ S^{2n-1} & \xrightarrow{\theta} & S^{2n-1} & \xrightarrow{\pi} & P^{2n-1} \end{array}$$

is commutative and there results a unique map f yielding commutativity in the second square; S^m, P^m, θ and π stand for the m-sphere,

the projective m-space, the antipodal map on S^m and the associated projection. This means that f induces an isomorphism of the "Deck-bewegungsgruppe" of $(\tilde{X}^*, p)$ onto that of (S^{2n-1}, π); by ([2], Lemme 1), it follows that

$$\operatorname{cat}(\tilde{X}^*, p) \leqslant \operatorname{cat}(S^{2n-1}, \pi)$$

and the latter is well known to equal $2n$.

By the definition of $\operatorname{cat}(\tilde{X}^*, p)$, (ii) yields $2n$ open subsets U_i of $X \times X$ such that

$$X \times X - \Delta = \bigcup_{i=1}^{2n} U_i \cup t(U_i) \quad \text{and} \quad U_i \cap t(U_i) = \emptyset .$$

Since $X \times X - \Delta$ is metrizable, hence normal, there exist open subsets V_i and W_i of $X \times X$ such that

$$X \times X - \Delta = \bigcup_{i=1}^{2n} V_i \cup W_i , \qquad \overline{V}_i - \Delta \subset U_i , \qquad \overline{W}_i - \Delta \subset t(U_i) .$$

The $2n+1$ subsets satisfying (iii) may now be obtained as

$$A_i = \overline{V}_i \cup t(\overline{W}_i) \quad \text{for} \quad i = 1, \ldots, 2n ,$$

$$A_{2n+1} = (\Delta \cup \bigcup_{i=1}^{2n} \overline{W}_i) - \bigcup_{i=1}^{2n} V_i \cup t(W_i) .$$

In order to derive (iv), notice first that, if A_i $(1 \leqslant i \leqslant 2n+1)$ satisfy (iii), then

$$X \times X = \bigcup_{i=1}^{2n+1} A_i \quad \text{and} \quad t(A_{2n+1}) \subset (X \times X - A_{2n+1}) \cup \Delta$$

imply $A_{2n+1} \subset t(A_1) \cup \ldots \cup t(A_{2n}) \cup \Delta$, hence

$$X \times X = \Delta \cup \bigcup_{i=1}^{2n} A_i \cup t(A_i) .$$

The product metric ϱ in $X \times X$ is invariant under t and the relations

$$g_i(x_1, x_2) = \varrho\big((x_1, x_2), A_i\big) - \varrho\big((x_1, x_2), t(A_i)\big) ,$$
$$g(x_1, x_2) = \big(g_1(x_1, x_2), \ldots, g_{2n}(x_1, x_2)\big) \in R^{2n}$$

define maps such that $x_1 \neq x_2$ implies

$$g_i(x_1, x_2) = -g_i(x_2, x_1) \neq 0$$

for some i, hence $g(x_1, x_2) \neq g(x_2, x_1)$.

Finally, if such a map $g\colon X\times X\to R^{2n}$ exists, putting

$$\tilde{f}(x_1, x_2) = \frac{g(x_2, x_1) - g(x_1, x_2)}{\|g(x_2, x_1) - g(x_1, x_2)\|} \in S^{2n-1}$$

for $x_1 \neq x_2$, defines a map $\tilde{f}$ for which the first square of (1) is commutative. There results a unique map f yielding commutativity in the second square and, introducing Wu Wen-Tsün's (singular) cohomology classes, by ([6]; (3)) we obtain

$$\Phi^{2n}(X) = A^{2n}(\tilde{X}^*, X^*, p) = f^* A^{2n}(S^{2n-1}, P^{2n-1}, \pi)\,.$$

The latter obviously vanishes and, since $n \neq 2$, by ([6]; Theorems 4 and 4') the vanishing of $\Phi^{2n}(X)$ implies that X can be imbedded in R^{2n}.

Remark 1. Actually, the statements (ii), (iii) and (iv) are all equivalent and implied by (i) whatever be $n \geqslant 1$.

Quasi-homeomorphic imbeddings

Let K be a finite simplicial complex. We write $\tilde{K}^*$ for the subcomplex of the cell-complex $K\times K$ which consists of all products of simplices of K having no common vertices. Identifying in $\tilde{K}^*$ every point (a_1, a_2) with (a_2, a_1) yields a new cell-complex K^* and a natural projection $\omega\colon |\tilde{K}^*|\to|K^*|$, where $|L|$ stands for the underlying space of any complex L.

Let $X = |K|$. Then $|\tilde{K}^*|$ and $|K^*|$ may be regarded as subspaces of $\tilde{X}^*$ and X^*, respectively, and, by ([6]; (6)), the inclusion map $i\colon |K^*|\to X^*$ induces isomorphisms of cohomology groups

$$i^*\colon H^m(X^*; I_{(m)}) \approx H^m(|K^*|; I_{(m)}) \tag{2}$$

in all dimensions; here, $I_{(m)}$ stands for the group of integers if m is even and for the group of integers mod. 2 if m is odd. Moreover, Wu Wen-Tsün's classes satisfy

$$A^m(|\tilde{K}^*|, |K^*|, \omega) = i^* A^m(\tilde{X}^*, X^*, p) \tag{3}$$

for all $m \geqslant 0$.

Let $\varepsilon > 0$; we say that a metric space X can be ε-mapped into a space E if there exists a map $f\colon X\to E$ such that $\operatorname{diam} f^{-1}(y) < \varepsilon$ for every $y \in f(X) \subset E$. Two metric spaces X and Y are quasi-homeomorphic if, for every $\varepsilon > 0$, each of them can be ε-mapped onto the other.

Theorem 2. *For every metric compact polyhedron X of dimension $n \neq 2$ there exists a positive $\varepsilon = \varepsilon(X)$ with the following property: if X can be ε-mapped into R^{2n}, then X can be topologically imbedded in R^{2n}.*

Proof. Let ϱ be the distance function induced in $X \times X$ by the metric of X, i. e.

$$\varrho^2((a, b), (c, d)) = \|a-c\|^2 + \|b-d\|^2,$$

where $\|x_1 - x_2\|$ is the distance between x_1 and x_2 in X. Let K be a simplicial complex covering the polyhedron X; consider the diagonal Δ of $X \times X$ and the compact subset $|\widetilde{K}^*|$ of $X \times X$, and let

$$\varepsilon = \varrho(|\widetilde{K}^*|, \Delta) = \varepsilon(X).$$

Since $|\widetilde{K}^*| \cap \Delta = \emptyset$, we have $\varepsilon > 0$. Suppose

$$f\colon X \to R^{2n}$$

satisfies $\operatorname{diam} f^{-1}(y) < \varepsilon$ for all $y \in Y$, where $Y = f(X) \subset R^{2n}$. Then $f(x_1) = f(x_2)$ implies

$$\varrho((x_1, x_2), (x_2, x_2)) = \|x_1 - x_2\| < \varepsilon \quad \text{and} \quad (x_1, x_2) \notin |\widetilde{K}^*|.$$

As a consequence, the relations

$$\widetilde{\psi}(x_1, x_2) = (f(x_1), f(x_2))$$
$$\psi\{x_1, x_2\} = \{f(x_1), f(x_2)\}$$

define maps for which the diagram

$$\begin{array}{ccccc} |\widetilde{K}^*| & \xrightarrow{\tau} & |\widetilde{K}^*| & \xrightarrow{\omega} & |K^*| \\ \downarrow{\scriptstyle \widetilde{\psi}} & & \downarrow{\scriptstyle \widetilde{\psi}} & & \downarrow{\scriptstyle \psi} \\ \widetilde{Y}^* & \xrightarrow{t'} & \widetilde{Y}^* & \xrightarrow{p'} & Y^* \end{array}$$

is commutative; as usual, τ and t' are the antipodal maps, ω and p' the natural projections. By ([6], (3)) we therefore have

$$A^m(|\widetilde{K}^*|, |K^*|, \omega) = \psi^* A^m(\widetilde{Y}^*, Y^*, p') \tag{4}$$

for all $m \geqslant 0$. Since Y lies in R^{2n}, by ([6]; Main Theorem) we obtain $A^{2n}(\widetilde{Y}^*, Y^*, p') = 0$; letting $m = 2n$ in (4) and (3) and taking into account that i^* in (2) is an isomorphism, we finally obtain

$$\Phi^{2n}(X) = A^{2n}(\widetilde{X}^*, X^*, p) = 0.$$

Since $n \neq 2$, by ([6], Theorems 4 and 4') this implies that X can be imbedded in R^{2n} as asserted.

Remark 2. We have actually proved that, whatever be $n \geqslant 1$, there exists $\varepsilon = \varepsilon(X) > 0$ such that if the n-dimensional polyhedron X can be ε-mapped into R^{2n}, then $\Phi^{2n}(X) = 0$.

For an n-dimensional complex the van Kampen [3] invariant coincides with Φ^{2n} (see [6]; p. 576); moreover, the van Kampen invariant

of the n-skeleton of a $(2n+2)$-simplex is non-vanishing ([3]; § 13)). We therefore have the

COROLLARY. *Marczewski's function $\varphi(n)$ equals $2n+1$ for every $n \geqslant 1$.*

Remark 3. Borsuk [1] has recently exhibited two compact quasi-homeomorphic spaces of which one is a subset of the plane, whereas the other cannot be imbedded in the plane. A universal generalization of Theorem 2, bearing on arbitrary n-dimensional compacta and Euclidean spaces of arbitrary dimension $m < 2n+1$, is therefore impossible.

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MATHEMATICS

On Involution in Banach Spaces

by
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Borsuk's classical theorem on antipodes [2] was extended by M. A. Krasnoselsky [5] to the case of completely continuous displacements in Banach spaces. On the other hand, A. I. Fet [3] proved a generalization of Borsuk's theorem on antipodes for the case of an n-dimensional Euclidean sphere, which consists in replacing the antipodal mapping by a continuous involution without a fixed point. This result was also obtained (independently) by J. W. Jaworowski [4] as a corollary of a more general argument.

In paper [1] a theorem was proved, which is a generalization of the above mentioned theorems, in the case of completely continuous displacement of the sphere in a Banach space.

The present note gives a more precise formulation of this theorem.

Denote by Q the full sphere defined in a Banach space X by the inequality $\|x\| \leqslant 1$, and by S the boundary of Q.

Let F be a completely continuous (generally non-linear) operator, defined on S with values in X, i. e. F be continuous and the image $F(S)$ of S under F be a compact set in X.

Consider the continuous operator

$$f = I - F,$$

defined on S, where I is the identical transformation of X. Suppose that

$$f(x) \neq 0 \quad \text{for every} \quad x \text{ of } S, \tag{1}$$

where 0 is the zero-element in X.

Further, let U be a continuous dimension non-rising involution of S onto itself without a fixed point, i. e. a continuous mapping U such that $U(U(x)) = x$ for every x of S. The term "dimension non-rising" means here that the image of an n-dimensional sphere under U lies in an n-dimensional one, for almost all positive integers n.

Then the following theorem holds:

THEOREM 1. *Let us assume that, on the boundary S of the sphere Q, a continuous mapping $f(x) = x - F(x)$ is defined, where F is a completely continuous operator with values in X, satisfying condition* (1). *Let U be a continuous dimension non-rising involution of S onto itself without a fixed point and satisfying the following condition*

$$\frac{f(Ux)}{\|f(Ux)\|} \neq \frac{f(x)}{\|f(x)\|} \tag{2}$$

for every x of S.

*Then the degree *) of f is odd.*

Proof. First consider the case when the mapping f satisfies condition

$$f(x) = -f(Ux) \tag{3}$$

for every x of S.

Following Leray and Schauder [6] we choose in X a finite-dimensional linear subspace E_n of X containing an ε-net of the compact set $F(S)$ as follows. Let E_p be the finite-dimensional linear space spanned by an ε-net of the set $F(S)$ and $x_1, x_2, \ldots, x_p$ a normed basis in E_p. Denote by E_n the linear space spanned by the elements $x_1, x_2, \ldots, x_p$ and $U(x_1), U(x_2), \ldots, U(x_p)$. Since the mapping U is dimension non-rising, the image $U(S^{(n-1)})$ of the sphere $S^{(n-1)} = S \cap E_n$ is contained, by hypothesis, in an $(n-1)$-dimensional sphere in X. But this sphere contains all elements x_i, $U(x_i)$ $(i = 1, 2, \ldots, p)$ and therefore it coincides with $S^{(n-1)}$. Thus, the mapping U is a homomorphism of $S^{(n-1)}$ into itself, whence we conclude that U is an involution of $S^{(n-1)}$ onto itself.

It is easy to see by (1) that there exists a positive number a such that

$$\|f(x)\| > a \quad \text{for every} \quad x \text{ of } S. \tag{4}$$

Let P be a continuous projection operator, defined on $F(S)$ with domain in E_n, such that

$$\|P_\varepsilon F(x) - F(x)\| < \varepsilon \quad \text{for} \quad x \text{ of } S, \tag{5}$$

where $\varepsilon < a/2$.

It is easy to see that operator P_ε can be so constructed that condition

$$P_\varepsilon(y) = -P_\varepsilon(-y) \tag{6}$$

will be satisfied for every y of $F(S)$. For this purpose it is enough to join the points $-y_1, -y_2, \ldots, -y_m$ to ε-net of $F(S)$ consisting of the points $y_1, y_2, \ldots, y_m$ and then construct the projection operator corresponding to the system of points $y_1, \ldots, y_m, -y_1, \ldots, -y_m$.

*) i. e. the degree $d[f \cdot Q, 0]$ in the sense of Leray-Schauder [6].

We now define by the formula

$$f_\varepsilon(x) = x - P_\varepsilon F(x) \quad \text{for} \quad x \in S^{(n-1)}$$

a continuous operator on $S^{(n-1)}$.

We shall show that the mapping $\frac{f_\varepsilon(x)}{\|f_\varepsilon(x)\|}$ satisfies the conditions of the theorem of Fet-Jaworowski [3], [4].

In fact, the values of $f_\varepsilon(x)$ on $S^{(n-1)}$ are different from the zero-element, since by (4) and (5) we have

$$\|f_\varepsilon(x)\| \geqslant \|x - F(x)\| - \|F_\varepsilon F(x) - F(x)\| > \frac{a}{2} \quad \text{for} \quad x \text{ of } S^{(n-1)}.$$

Further, by (3) and (6), we have

$$f_\varepsilon(x) = -f_\varepsilon(Ux) \quad \text{for} \quad x \text{ of } S^{(n-1)}.$$

Using the above mentioned result of Fet-Jaworowski we infer that the degree of the mapping $f(x)/\|f_\varepsilon(x)\|$ is odd. Finally, following Leray-Schauder [6], we can state that the degree of the mapping f is odd and thus the theorem is proved for the case when f satisfies condition (3).

To prove the theorem generally, consider the family of transformations

$$f_t(x) = \frac{1}{1+t} f(x) - \frac{t}{1+t} f(Ux) \quad (x \in S;\ 0 \leqslant t \leqslant 1). \tag{7}$$

f_t is of the form

$$f_t(x) = x - G_t(x),$$

where $G_t(x) = \frac{1}{1+t} F(x) - \frac{t}{1+t} F(Ux)$.

For any t, G_t is a completely continuous transformation. G_t is uniformly continuous in t, since

$$\|G_{t'}(x) - G_{t''}(x)\| \leqslant 2|t' - t''| \cdot |\sup_{y \in S} \|F(y)\|, \quad (x \in S).$$

$f_t(x)$ does not vanish on S. In fact, if we have $f_{t_0}(x_0) = 0$ for x_0 and t_0, then

$$f(x_0) = t_0 f(Ux_0)$$

by (7) which is a contradiction of (2).

Thus, transformations $f_0(x)$ and $f_1(x)$ are homotopic. But $f_0(x) = f(x)$ and

$$f_1(x) = \tfrac{1}{2} f(x) - \tfrac{1}{2} f(Ux)$$

satisfies condition (3). Hence, the degrees of mappings f and f_1 are equal and the theorem is proved in the general case.

Theorem 1 implies the following

THEOREM 2. *Let F be a completely continuous operator defined on the sphere Q with values in X without fixed point on the boundary S. If con-*

dition (2) *is satisfied, then* F *possesses a fixed point in* Q, *i. e. there exists an element* x_0 *of* Q *such that*

$$F(x_0) = x_0 .$$

Remark 1. As a particular case of the above theorems we obtain Theorems 1 and 2, proved in [1], provided that condition (2) is replaced by condition

$$\inf_{x \in S} \left\| \frac{f(x)}{\|f(x)\|} - \frac{f(Ux)}{\|f(Ux)\|} \right\| > 0 . \tag{8}$$

If the involution U is replaced by the antipodal mapping, i. e. $U(x) = -x$ for x of S, then conditions (2) and (8) are equivalent and we obtain Krasnoselsky's extension to Banach spaces of Borsuk's theorem on antipodes.

Remark 2. Notice that, if F is a completely continuous linear operator in the Banach space X, then condition (2) or (3) is evidently satisfied for $U(x) = -x$. Condition (1) is satisfied if and only if the equation

$$f(x) = x - F(x) = 0$$

has only a zero solution. In this case, equation

$$x - F(x) = y$$

has a solution for any y of X which follows immediately from the theorem on antipodes in Banach space.

We thus see that this part of the "Fredholm alternative" is a trivial consequence of the theorem on antipodes in Banach space.

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MATHEMATICS

Continuous Transformations of a Locally Convex Space onto Itself

by

M. ALTMAN

Presented by K. BORSUK on November 10, 1958

Paper [1] concerns completely continuous displacements of open sets in locally convex linear topological spaces. In the present note we consider a class of locally completely continuous displacements of a locally convex linear topological space onto itself. For this purpose we introduce the notion of a ϑ-mapping in the narrow sense, which is the generalization of the notion of an ε-mapping in the narrow sense [3]. The results obtained in the present note contain, in particular, Borsuk's theorem on ε-mappings in the narrow sense [3] and the corresponding theorems proved by A. Granas [4], [5] for Banach spaces. We also consider here the case of a locally completely continuous displacement in an open mapping. The proofs of the theorems given in this note are based on the results of paper [1].

1. Let X be a locally convex linear topological space, i. e. a linear Hausdorff space with the topology defined by means of a system of neighbourhoods of origin 0, which are symmetric convex open sets.

Let $U(x) = x \oplus U$ be a neighbourhood of x, where U is an open convex symmetric neighbourhood of the origin and $x \oplus U$ is the direct sum of x and U, i. e. the set of all elements $x+y$ for fixed x and arbitrary y of U. Denote by $\overline{U}$ the closure of U. The symbol ϑU denotes the boundary of U, i. e. $\vartheta U = \overline{U} - U$.

A transformation f defined on $\overline{U}(x)$ with values in X is called a ϑ-mapping on $\overline{U}(x)$ in the narrow sense if there exists an open convex symmetric neighbourhood V of the origin satisfying the following condition:

$$x', x'' \in \overline{U}(x) \quad \text{and} \quad f(x') - f(x'') \in V \quad \text{imply} \quad x' - x'' \text{ non} \in \vartheta U .$$

A continuous transformation F defined on X with values in X is said to be ***locally completely continuous*** if for any x of X there exists

an open convex symmetric neighbourhood U_x of the origin such that the image $F(\overline{U}_x(x))$ of the closure of $U_x(x) = x \oplus U_x$ under F is a subset of a bicompact set in X.

A continuous transformation f is said to be a *locally completely continuous displacement* if f is of the form

$$f(x) = x - F(x), \quad x \in X, \tag{1}$$

where F is a locally completely continuous mapping with values in X.

THEOREM 1. *Let f be a locally completely continuous displacement of the form* (1) *being also locally a ϑ-mapping in the narrow sense, which means that there exists an open convex symmetric neighbourhood V of such origin that, for every x of X, there exists an open convex symmetric neighbourhood U_x of the origin satisfying the following conditions:*

1^0 *the image $F(\overline{U}_x(x))$ of the closure of $U_x(x) = x \oplus U_x$ under F is a bicompact subset in X;*

2^0 *$x', x'' \in \overline{U}_x(x)$ and $f(x') - f(x'') \in V$ imply $x' - x''$ non $\in \vartheta U_x$.*

3^0 *U_x does not contain the whole straight line $\{tf(x)\}$ if $f(x) \neq 0$.*

Then f maps X onto the whole X.

Proof. Let x be an arbitrary element of X. It follows from 2^0 that f is a ϑ-mapping on $\overline{U}_x(x)$, i. e.

$$x', x'' \in U_x(x) \text{ and } f(x') = f(x'') \text{ imply } x' - x'' \text{ non} \in \vartheta U_x,$$

and the intersection $f(\vartheta U_x(x)) \cap (f(x) \oplus V)$ is empty. Using the same argument as in the proof of Theorem 1 in [1] we infer that the degree $d[f, U_x(x), f(x)]$ is odd. Thus, image $f(U_x(x))$ contains the neighbourhood $f(x) \oplus V$ of $f(x)$. Since x is an arbitrary element of X and V is a fixed neighbourhood independent of x, the theorem is proved.

Theorem 1 contains, in particular, the corresponding theorems proved by A. Granas [4], [5] for a Banach space. Theorem 1 is a generalization of Borsuk's theorem on ε-mappings in the narrow sense [3] for a finite-dimensional Euclidean space.

2. A continuous transformation of the form (1) is said to be *strictly locally a ϑ-mapping in the narrow sense* if there exists a fixed open convex symmetric neighbourhood V of an origin with the following property:

(B) for any x of X and for an arbitrary neighbourhood $W(x)$ of x there exists an open convex symmetric neighbourhood U_x of the origin such that

the closure $\overline{U}(x)$ of $U(x) = x \oplus U_x$ is contained in $W(x)$ and conditions 1^0-3^0 are satisfied.

By means of this notion we can consider the case when f is an open mapping onto X.

THEOREM 2. *Let f be a locally completely continuous displacement of the form* (1) *with values in X. If f is strictly locally a ϑ-mapping in the narrow sense, i. e. if condition* (B) *is satisfied, then f is an open mapping onto X.*

This theorem is contained in [2] provided that X is a Banach space.

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An Approximation Method for Solving Scattering Problems

by

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1. We shall discuss in this paper a relativistic approximation method for solving scattering problems. This method can be useful in considering processes involving strongly interacting particles, where the use of a perturbation method is forbidden by the high value of the coupling constant. For the sake of simplicity, we shall limit our considerations to the spinor (isospinor) field ψ_2 interacting with a pseudoscalar (isovector) field φ^i, the Hamiltonian interaction being $ig \sum_k \bar{\psi}\gamma^5\tau^k\psi\varphi^k$. The generalisation to other interactions is obvious. We shall work in the Heisenberg representation. In this representation the field operators satisfy the equations

$$(\partial_\mu\gamma^\mu + m)\psi = ig \sum_k \gamma^5\tau^k\psi\varphi^k\,,$$

$$(\Box - \mu^2)\varphi^i = ig\bar{\psi}\gamma^5\tau^i\psi\,. \tag{1}$$

Let us introduce the asymptotic operators $\psi^{(\mathrm{in})}$, $\varphi^{i(\mathrm{in})}$ ($\psi^{(\mathrm{out})}$, $\varphi^{i(\mathrm{out})}$) which pass into ψ and φ^i for $t \to -\infty(+\infty)$. These are the same operators which have been defined by Lehmann, Symanzik and Zimmermann in a series of papers concerning the relativistic S-formalism in the Heisenberg representation. As in the latter papers we limit ourselves to scattering problems and do not consider the bound states. For the generalisation of results of the quoted authors to the problems involving the bound states see [4]. The operators $\psi^{(\mathrm{in})}$, $\varphi^{i(\mathrm{in})}$, ($\psi^{(\mathrm{out})}$, $\varphi^{i(\mathrm{out})}$) satisfy the free particle equations

$$(\partial_\mu\gamma^\mu + m)\psi^{(\mathrm{in})} = 0\,,$$

$$(\Box - \mu^2)\varphi^{i(\mathrm{in})} = 0$$

(it is essential that the masses appearing there be physical ones), and the respective commutation relations are:

$$[\psi^{(\mathrm{in})}(x), \bar{\psi}^{(\mathrm{in})}(x')]_+ = iS(x-x')$$

$$[\varphi^{i(\mathrm{in})}(x), \varphi^{k(\mathrm{in})}(x')]_- = i\delta_{ik}\Delta(x-x')\,.$$

We write Eqs. (1) in the integral form:

$$\psi(x) = \psi^{(\mathrm{in})}(x) + ig \sum_k \int S^{\mathrm{ret}}(x-x_1)\gamma^5\tau^k\psi(x_1)\varphi^k(x_1)\,d_4x_1$$

(2)

$$\varphi^i(x) = \varphi^{i(\mathrm{in})} + ig \int \Delta^{\mathrm{ret}}(x-x_1)\bar{\psi}(x_1)\gamma^5\tau^i\psi(x_1)\,d_4x_1$$

and

$$\psi(x) = \psi^{(\mathrm{out})}(x) + ig \sum_k \int S^{\mathrm{adv}}(x-x_1)\gamma^5\tau^k\psi(x_1)\varphi^k(x_1)\,d_4x_1$$

(3)

$$\varphi^i(x) = \varphi^{i(\mathrm{out})} + ig \int \Delta^{\mathrm{adv}}(x-x_1)\bar{\psi}(x_1)\gamma^5\tau^i\psi(x_1)\,d_4x_1\,.$$

In the case of a small coupling constant we can solve Eqs. (2) by iteration – the terms proportional to the higher powers of g are then negligible. In the case of a large coupling constant this iteration expansion has only a formal significance. Formally, ψ and φ are represented as the infinite sum of integrals of products of $\psi^{(\mathrm{in})}$, $\varphi^{(\mathrm{in})}$ and respective Green functions. However, the operator character of function $\psi^{(\mathrm{in})}$ and $\varphi^{(\mathrm{in})}$ makes it possible to present ψ and φ as the infinite sum of terms involving such combinations of operators $\psi^{(\mathrm{in})}$, $\varphi^{i(\mathrm{in})}$ that the form of coefficients near those combinations independently of the magnitude of g ensures a real significance to our expansion. The approximation consists in cutting off the higher terms of expansion. We find the remaining coefficients by solving a set of integral equations. According to the expansion employed, we have distinct approximations. Unfortunately, the mathematical investigation of the degree of convergence of such an approximation is a very difficult problem, similarly as in the case of the as yet open question of the convergence of the Tamm-Dancoff method. The scope of the applicability of the given approximation must then be fixed by a comparison of the numerical and experimental results obtained. Depending on the use in our expansion of "in" or "out" operators, we obtain two approximations, which converge to the same limit and are called further retarded and advanced, respectively. In principle we can expand ψ and φ using any operators which satisfy the free field equations and commutation relations, e. g. we can express the field operators in the Heisenberg representation by the operators in the interaction representation. However, a difficulty would then appear owing to the fact that the vacuum defined by the latter operator is not the physical vacuum (in our case we have $|\mathrm{vac}\rangle_{\mathrm{in}} = |\mathrm{vac}\rangle_{\mathrm{out}} = |\mathrm{vac}\rangle_{\mathrm{phys}}$) and on the basis of our expansion it would be difficult to find the probabilities for concrete physical processes.

2. In what follows we shall limit ourselves to the one particular kind of expansion and to the retarded approximation. We obtain the expression for the advanced approximation in the same way. Let us write

$$\psi^{(\mathrm{in})} = \psi^{(\mathrm{in})+} + \psi^{(\mathrm{in})-},\quad \varphi^{i(\mathrm{in})} = \varphi^{i(\mathrm{in})+} + \varphi^{i(\mathrm{in})-}\,.$$

Let us commute in our formal iteration expansion all annihilation operators to the right, and let us group all the expressions having the same operator structure, i. e. those which contain an equal number of the same creation and annihilation operators. We then obtain, as already found by Glaser, Lehmann and Zimmermann [3], the expansion of operators ψ, φ on the normal products of asymptotic operators:

$$\begin{aligned}\psi(x) &= \int K_1(x-x_1)\psi^{(\mathrm{in})}(x_1)\,d_4x_1 + \\ &\quad + \sum_k \int K_2^k(x-x_1,\, x-x_2)\psi^{(\mathrm{in})}(x_1)\varphi^{k(\mathrm{in})}(x_2)\,d_4x_1\,d_4x_2 + \ldots \\ \varphi^i(x) &= \sum_k \int H_1^{ik}(x-x_1)\varphi^{k(\mathrm{in})}(x_1)\,d_4x_1 + \ldots\end{aligned} \tag{4}$$

Using creation operators "in" and vacuum state $|\mathrm{vac}\rangle_{\mathrm{in}}$ we can construct a basis in the Hilbert space. Substituting the expansion (4) into Eqs. (2) and taking the matrix elements of both sides of the equations obtained in this way between all state vectors of our basis we obtain an infinite set of integral equations. If we put $K_n = 0$ for $n > N$ and $H_m = 0$ for $m > M$, we obtain a finite set of equations and we can begin to solve them. At this point we must note that the omission of higher terms of expansion yields the approximate expression for the field operators. The commutation (anticommutation) relations existing in the exact theory will, in general, be preserved only for $N, M \to \infty$. If we do not want to obtain in our final formulae the accidental terms, we must substitute a symmetrized (antisymmetrized) expression in each place where we have the product of commuting (anticommuting) in the exact theory operators, e. g. where a product $\psi(x)\varphi(x)$ occurs in exact equations; passing to the approximation we put the expression $\frac{1}{2}(\psi\varphi + \varphi\psi)$. The solution thus obtained passes into the exact solution when $N, M \to \infty$ but the discretion which we should have choosing $\varphi\psi$ or $\psi\varphi$ does not appear. We have *a priori* two principal possibilities of applying our result after solving integral equations for coefficients K and H. We can calculate directly the elements of S-matrix, i. e. quantities ${}_{\mathrm{in}}\langle nk|n'k'\rangle_{\mathrm{out}}$, where $|nk\rangle_{\mathrm{in}}$ is an asymptotic state for $t \to -\infty$ representing n fermions and k besons, and $|n'k'\rangle_{\mathrm{out}}$ is an asymptotic state for $t \to \infty$ representing n' fermions and k' bosons. Vector $|nk\rangle_{\mathrm{in}}(|n'k'\rangle_{\mathrm{out}})$ is constructed by using "in" ("out") creation operators. We can find ${}_{\mathrm{in}}\langle nk|n'k'\rangle_{\mathrm{out}}$ after expressing the operators "out" by "in", because we know the commutation relations between "in" operators.

However, it is difficult to do this so that the accuracy of calculation be symmetrical with respect to "in" and "out" operators. On the other hand, the unsymmetrical calculation gives doubtful results. Let us express ψ, φ in an approximate manner by $\psi^{(\mathrm{in})}, \varphi^{(\mathrm{in})}$ but $\psi^{(\mathrm{out})}\varphi^{(\mathrm{out})}$

accurately by ψ, φ using Eqs. (3). Writing

$$\psi^{(\mathrm{out})}(x) = \psi(x) - ig \sum_k \int S^{\mathrm{adv}}(x - x_1)\gamma^5 \tau^k \psi(x_1)\varphi^k(x_1)\, d_4 x_1$$

$$\varphi^{i(\mathrm{out})}(x) = \varphi^i - ig \int \Delta^{\mathrm{adv}}(x - x_1)\bar{\psi}(x_1)\gamma^5 \tau^i \psi(x_1)\, d_4 x_1$$

and calculating the S-matrix element corresponding to the boson fermion scattering we obtain, by using the simplest approximation discussed further on, a result independent of the coupling constant, which is inadmissible on physical ground. The second possibility gives more attractive results consisting in a calculation of the propagators (e. g. $\langle \mathrm{vac}|T(\psi_\alpha \bar{\psi}_\beta)|\mathrm{vac}\rangle$) by using retarded (advanced) approximations for ψ, φ.

3. Let us look for a non-trivial solution of the simplest approximate equations. If we limit ourselves to the first terms of expansions (4) we obtain the trivial result $\psi = \psi^{(\mathrm{in})}$, $\varphi = \varphi^{(\mathrm{in})}$. If we cut off the expansion of ψ after the second term, and the expansion of φ after the first term, we obtain for our coefficients nonlinear equations difficult to solve effectively. Let us limit our approximation as follows: let us put $\varphi^{(\mathrm{in})} = \varphi^{(\mathrm{out})} = \varphi$ and let us expand only the operator ψ. Physically this means neglecting the virtual and real fermion pair production by bosons. We obtain the following equations putting $K_n = 0$ for $n > 2$:

$$K_1(x - x') = \delta(x - x')$$

$$(5) \qquad - \sum_k \frac{g}{2} \int S^{\mathrm{ret}}(x - x_1)\gamma^5 \tau^k K_2^k(x_1 - x', x_1 - x'')\Delta'(x'' - x_1)\, d_4 x_1\, d_4 x''$$

$$K_2^i(x - x', x - x'') = ig\, S^{\mathrm{ret}}(x - x'')\gamma^5 \tau^i K_1(x'' - x')\,.$$

We pass to the momentum representation and solve our equations

$$K_1(p) = \left(I + \frac{3ig^2(2\pi)^8}{2} S^{\mathrm{ret}}(p) \int \gamma^5 S^{\mathrm{ret}}(p + s)\gamma^5 \Delta^1(s)\, d_4 s\right)^{-1} = A \cdot I + B p_\mu \gamma^\mu$$

$$K_2^i(p, k) = ig\, S^{\mathrm{ret}}(p + k)\gamma^5 \tau_i (A \cdot I + B p_\mu \gamma^\mu)\,.$$

The quantities A and B are functions of p^2, g, m, μ, and M – an auxiliary mass which appear as a consequence of regularization of the function Δ^1 by the Pauli-Villars method, for the purpose of removing the divergence of the integral $\int S^{\mathrm{ret}}(p + s)\Delta^1(s)\, d_4 s$.

We see that our method is a cut-off method but the cut-off is introduced in a covariant way, in contradiction to the other approximations (Chew) used in the strong interactions theory. A and B are meromorphic functions of g^2, the denominators being polynomials of the second and numerators of the first order relatively to g^2. We do not write here explicitely A and B. This will be done in a later paper, where the numerical results of our approximation will be compared with experiment.

Knowing K_1 and K_2 we can calculate the propagator

$$S'_{F_{\alpha\beta}}(x-x') = \langle \mathrm{vac}|T(\psi_\alpha(x)\bar{\psi}_\beta(x'))|\mathrm{vac}\rangle$$

$$S'_F(p) = K_1(p)\,S_F(p)\,\bar{K}_1(p) + \sum_k \int K_2^k(p-s,\,s)\,S_F(p-s)\,\bar{K}_2^k(p-s,\,s)\,\Delta_F(s)\,d_4 s,$$

where

$$\bar{K}_1 = \gamma^4 K_1^* \gamma^4, \quad \bar{K}_2 = \gamma^4 K_2^* \gamma^4.$$

Using our approximate expressions for ψ, φ we can calculate the many-particle propagators. However, the result may be trivial if the number of particles is sufficiently high, e. g. such are, in the simplest approximation, the propagators describing the boson production.

By using the calculated propagators we can find the S-matrix elements corresponding to the more complicated processes in a semi-perturbative way (this may be useful in the case of photoproduction, e. g. in calculating the shaking-off transition element using the propagator S'_F). We can also tackle a problem, whose importance for examining the symmetries of weak interactions was pointed out by B. d'Espagnat and J. Prentki: it consists in calculating the corrections to the transitions generated by weak interactions when the strong interactions are taken into account. We notice here the essential property of the proposed approximation. In the methods used thus far we look for the approximate equations of the amplitudes of states. In order to find these equations and in order that they be solvable we take out some amplitudes and equal to zero all others. In our method we look for the approximate expressions for field operators. Mathematically this is also equivalent to the solution of a set of integral equations, but the solutions found may be differently applied, e. g. in our simplest approximation we can write the S-matrix element for boson-fermion as well as for fermion-fermion scattering. We cannot say at present, on physical grounds, whether the approximation applicable to one process is also applicable to the other.

Finally, I should like to offer my sincere thanks to Królikowski D. S. for guidance and to the other members of the Warsaw theoretical group for many interesting remarks.

Summary

A new relativistic approximation method is proposed, which consists in using the approximate expression for interacting field operators in the Heisenberg representation.

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EXPERIMENTAL PHYSICS

Emission and Absorption Spectra of Molecules of Acridine Yellow in the Preexcited State

by
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Presented by A. JABŁOŃSKI on October 13, 1958

The present investigation is intended to provide an answer to the question, whether the preexcited state of acridine yellow in gelatine involves modifications in the spectra of absorption and of phosphorescence emission.

Investigation of the decay curves [1] of this organophosphor had shown that prolonged preexcitation modifies the probability of electronic transition from the metastable level to the fluorescence level in the luminescent centres. To explain the phenomenon, it was assumed that, during preexcitation, luminescent non-stable isomers of acridine yellow are formed.

In the present investigation, the same specimens of organophosphor were used as when investigating the decay curves [1]. The phosphors were preexcited continuously with natural light, for periods of 1, 2, 5, 10, 20 and 30 minutes. An interval of 1 minute separated the cessation of preexcitation and the beginning of investigation of the phosphorescence or absorption. The exposure of emission spectra lasted 30 minutes, the absorption measurements — 15 minutes. Photographs of the spectra were obtained in natural light, using a glass "Barfit" Hilger spectrograph for emission, and a Zeiss spectrophotometer for absorption measurements; the absorption and emission of phosphorescence were observed at 20°C, whilst the slow fluorescence spectra were obtained at the temperature of liquid air.

Investigation proved that variation of the period of preexcitation modifies all the spectra considered, i. e. the absorption and phosphorescence spectra and those of slow fluorescence. Practically, in the actual conditions, preexcitation for a period of 10 minutes resulted in saturation. Continued preexcitation caused no further modification. Fig. 1 shows the spectra of absorption, phosphorescence and slow fluorescence for 0,10 and 20 minutes preexcitation; it will be seen that absorption intensity rises in the preexcited state and that the maximum is shifted towards longer wavelenghts. The modifications of absorption in the

preexcited state are accompanied by a decrease in the phosphorescence intensity and by a shift of its maximum towards shorter wavelengths, as well as by a decrease in the slow fluorescence intensity, its maximum retaining its former position.

The positions of the maxima of the respective bands, the phenomena of phosphorescence and slow fluorescence in the non-preexcited

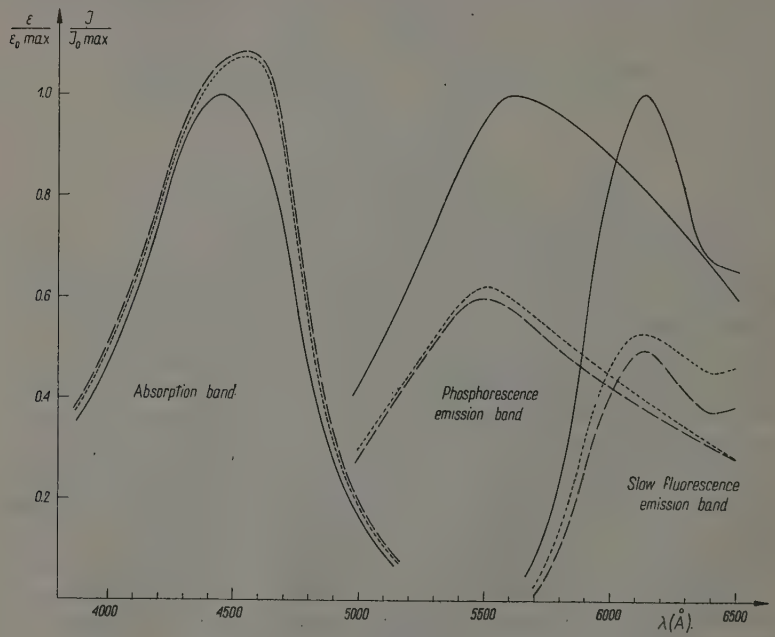

Fig. 1.
0 minutes preexcitation
10 " "
20 " "

state and in the "saturated" preexcited state may be explained by Jabłoński's diagram of electronic levels [2] (Fig. 2).

In our case, the observed spectra are not pure spectra of the non-preexcited molecules. Due to a rather prolonged excitation they constitute spectra of a mixture of the non-preexcited and of the preexcited molecules. On the other hand, the "saturated" preexcited state should be considered as greatly enriched in deformed molecules. Hence, it may be affirmed that the set of electronic levels of the non-deformed molecule of acridine yellow is related to the system of levels of Fig. 2 (a), whereas the deformed molecule presents a set of levels resembling that of Fig. 2 (b). However, no doubt may subsist as to the fact that the molecules in the non-deformed and in the deformed states possess different sets of energy levels.

As already mentioned, investigations by Frąckowiak and Heldt [1] had proved that the probability of the $M \to F$ electronic transition increases in the preexcited state with longer periods of excitation. Hence, the phosphor in the preexcited state must be enriched in deformed luminescent molecules whose probability Γ_{MF} for the $M \to F$ electronic transition exceeds the respective probability Γ'_{MF} of the non-deformed molecules. Let E_F, E_M, $E_{F'}$ and $E_{M'}$ denote, respectively, the energy of the fluorescence and metastable levels of the non-deformed and deformed molecules; then the present investigation leads to the conclusion that $E_F - E_M < E_{F'} - E_{M'}$. Considering Jabłoński's formula [2] for the $M \to F$ transition probability:

$$(1) \quad \Gamma = S_0 \exp\left(-\frac{E_F - E_M}{kT}\right)$$

with S_0 denoting a proportionality factor, k Boltzmann's constant and T the absolute Kelvin temperature, and the fact that $\Gamma' > \Gamma$, where Γ' is the $M' \to F'$ transition probability in preexcited molecule [1], it is found that $S_0 < S'_0$, wherein S_0 and S'_0 are the proportionality factors for the non-deformed and deformed molecule, respectively. Hence, it may be concluded that the constant S_0 in Eq. (1) is related with the electronic structure of the molecule. For this reason, the constant S_0 appearing in Eq. (1) may be termed the structural constant.

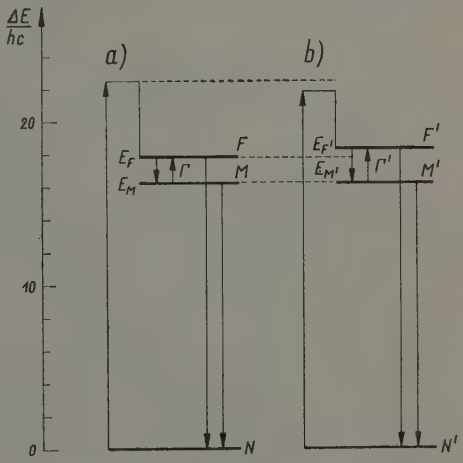

Fig. 2. Jabłoński's level diagram for a) non-preexcited molecule and b) preexcited molecule

It may be concluded from the present discussion that preexcitation of the molecules of acridine yellow leads to the formation of unstable, deformed isomers of molecules having a different set of energy levels from that of the non-deformed (original) molecules, and a different structural constant.

The author is indebted to Mr. M. Frąckowiak for his advice and for the interest shown in the present investigation.

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БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК

СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ И ФИЗИЧЕСКИХ НАУК

Резюме статей

ТОМ VII 1959 ВЫПУСК 1

Р. БИТТНЕР, НЕКОТОРАЯ АКСИОМАТИКА ДЛЯ ОПЕРАЦИОННОГО ИСЧИСЛЕНИЯ . . . . . стр. 1 – 9

Пусть S – аддитивная и однородная операция, определенная в линейном подпространстве Y линейного пространства X, причем $S(Y) = X$.

Пусть T – операция со свойством $STx = x$. Элемент $x - TSx = \gamma x$ принадлежит к ядру операции S. Линейную операцию R называем генератором, если выполняется условие $x - TRx = 0 \supset x = 0$. В изложенной работе понятие генератора использовано для построения операционного исчисления в линейных пространствах. В частности, дается анализ операторови формулировка теорем алгебры задаваемыя уравнениями $A_n S^n x + A_{n-1} S^{n-1} x + \ldots + A_0 x = f$, $\gamma S^k x = c_k$, $k = 0, 1, \ldots, n-1$, приводимыми к виду

$$(S - R_1)^{k_1} (S - R_2)^{k_2} \ldots (S - R_m)^{k_m} x = f, \quad \gamma S^k x = c_k,$$

$k = 0, 1, \ldots, n-1$, где $R_1, \ldots, R_m$ являются генераторами.

Г. АЛЬБРЫХТ, НЕКОТОРЫЕ ЗАМЕЧАНИЯ О ПРОСТРАНСТВЕ МАРЦИНКЕВИЧА-ОРЛИЧА. II. . . . . стр. 11–12

В настоящем сообщении приводятся основные свойства пространства $\mathcal{B}$, которое содержится в пространстве $\mathcal{O}$ Марцинкевича-Орлича.

Пространство $\mathcal{B}$ составлено из таких элементов пространства $\mathcal{O}$, которые в качестве представителя имеют некоторый тригонометрический полином. Оказывается, что пространство $\mathcal{B}$ является несепарабельным пространством Банаха.

Г. ЛОСЬ, ЛИНЕЙНЫЕ УРАВНЕНИЯ И СЕРВАНТНЫЕ ГРУППЫ стр. 13–18

Подгруппу G группы H называем $\varkappa$-сервантной в G, если каждая система линейных уравнений

$$(U) \qquad \sum_{s \in T} a_{t,s} x_s = g_t \in G, \quad t \in T$$

мощности не превышающей $\aleph_\varkappa$ и имеющая решение в H, имеет также решение в G.

$a_{t,s}$ — целые числа, причем при фиксированном t имеем $a_{t,s} \neq 0$ лишь для конечного числа s из множества T. Мощностью системы (U) называем мощность множества T.

Пусть S_λ^* — группа всех последовательностей целых чисел типа ω_λ. Для x, принадлежащего к S_λ^* полагаем $\nu(x) = \{\xi < \omega_\lambda;\ x_\xi \neq 0\}$, а затем $S_\lambda^\varkappa = \{x \in S_\lambda^*;\ |\nu(x)| < \aleph_\varkappa\}$.

Теорема 1. *Если мощность $\aleph_\varkappa$ регулярна, то группа $S_\lambda^\varkappa$ $\varkappa$-сервантна в S_λ^*.*

$\aleph_\varkappa$ регулярно, если $|T| < \aleph_\varkappa$ и $\mathfrak{m}_t < \aleph_\varkappa$ влечет за собой $\sum_{t \in T} \mathfrak{m}_t < \aleph_\varkappa$. Если $\varkappa = \mu + 1$, то $\aleph_\varkappa$ регулярно.

Группу G называем регулярной, если для любого элемента $g \neq 0$, $g \in G$ существует гомоморфизм h, отображающий G в группу целых чисел и такой, что $h(g) \neq 0$. Группу G называем вполне нерегулярной, если не существует гомоморфизм отображающий G на группу целых чисел.

Теорема 2. *Если мощность $\aleph_\varkappa$ меры нуль, а мощность $\aleph_\varkappa$ регулярна, то каждая подгруппа группы $S_\lambda^\varkappa = S_\lambda^*/S_\lambda^\varkappa$ мощности не превышающей $\aleph_\varkappa$ — регулярна, вся же группа $S_\lambda^\varkappa$ вполне нерегулярна.*

$\aleph_\lambda$ меры нуль, если в множестве T мощности $\aleph_\lambda$ не существует σ-мера со значениями 0 и 1, определенная для всех подмножеств множества T и равная нулю на одноточечных множествах.

Теорема 3. *Если мощность $\aleph_\varkappa$ регулярна, то каждая сервантная (0-сервантная) циклическая подгруппа группы $S_\lambda^\varkappa$ $\varkappa$-сервантна.*

Теорема 4. *Если мощность $\aleph_\varkappa$ регулярна и меры нуль, то декартово произведение $\aleph_\mu = 2^{\aleph_\varkappa}$ метрических не компактных пространств — не является $\varkappa$-компактным.*

Г. ЛОСЬ, ОБОБЩЕННЫЕ ПРЕДЕЛЫ В АЛГЕБРАИЧЕСКИ КОМПАКТНЫХ ГРУППАХ . . . . . стр. 19—21

Через λ — предел в абелевой группе G понимаем функционал, приписывающий каждой последовательности $\{x_\xi\}$ типа ω_λ элементов группы G элемент $\operatorname{Lim}_{\xi<\omega_\lambda} x_\xi$ также принадлежащий к группе G так, что выполняются следующие условия:

(I) $\operatorname{Lim}_{\xi<\omega_\lambda}(x_\xi + y_\xi) = \operatorname{Lim}_{\xi<\omega_\lambda} x_\xi + \operatorname{Lim}_{\xi<\omega_\lambda} y_\xi$;

(II) Если $x_\xi = g$ при $\xi < \omega_\lambda$, то $\operatorname{Lim}_{\xi<\omega_\lambda} x_\xi = g$;

(III) Если существует такое $\xi_0 < \omega_\lambda$, что $x_\xi = y_\xi$ при $\xi > \xi_0$, то $\operatorname{Lim}_{\xi<\omega_\lambda} x_\xi = \operatorname{Lim}_{\xi<\omega_\lambda} y_\xi$.

Теорема. *В группе G при каждом λ существует λ-предел тогда и только тогда, когда G — алгебраически компактная группа, т. е. когда G — прямое слагаемое такой группы H, в которую может быть введена компактная топология.*

Э. СОНСЯДА, ПОСТРОЕНИЕ ПРЯМО НЕРАЗЛОЖИМЫХ АБЕЛЕВЫХ ГРУПП МОЩНОСТИ БОЛЬШЕЙ, ЧЕМ КОНТИНУУМ. (II). . . стр. 23–26

В предшествующей работе [3] автор построил прямо неразложимую абелеву группу мощности $2^{\mathfrak{c}}$. Другие примеры прямо неразложимых абелевых групп мощности $2^{\mathfrak{c}}$ нашли Фукс [1], де Грот (еще неопубликованный пример) и Гуляницкий [2].

В представленной работе автор доказывает, что для $\aleph_1$ различных мощностей $\mathfrak{n}$ существуют прямо неразложимые абелевы группы мощности $\mathfrak{n}$. Более подробно: пусть $\mathfrak{n}_0 < \ldots < \mathfrak{n}_\alpha < \mathfrak{n}_{\alpha+1} < \ldots (\alpha < \omega_1)$ будет последовательностью мощностей таких, что $\mathfrak{n}_0 = 1$, $\mathfrak{n}_{\alpha+1} = 2^{\mathfrak{n}_\alpha}$, а для предельного порядкового числа α мощность $\mathfrak{n}_\alpha$ есть сумма всех предшествующих мощностей.

Теорема. *Для всякой мощности* $\mathfrak{n}_\alpha (\alpha < \omega_1)$ *существует прямо неразложимая абелева группа* G_α *ранга* $\mathfrak{n}_\alpha$.

Т. ГАНЕА, ЗАМЕТКА О ПОГРУЖЕНИИ ПОЛИЭДРОВ В ЭВКЛИДОВЫХ ПРОСТРАНСТВАХ . . . . . стр. 27–32

Опираясь на результаты, полученные Ву-Вен-Тсуном, касающиеся погружения полиэдров размерности $n \neq 2$ в эвклидовом $2n$-мерном пространстве R^{2n}, приводятся новые равносильные условия для возможности такого погружения. Кроме того, доказано, что для каждого метрического компактного полиэдра X размерности $n \neq 2$ существует $\varepsilon = \varepsilon(X) > 0$ такое, что если X может быть отображено в R^{2n} через ε-непрерывные отображения, то X гомеоморфно подмножеству R^{2n}.

М. АЛЬТМАН, ОБ ИНВОЛЮЦИЯХ В ПРОСТРАНСТВАХ БАНАХА . . . . . стр. 33–36

Пусть Q обозначает единичный шар, определенный в пространстве Банаха X неравенством $\|x\| \leqslant 1$ и пусть S обозначает границу шара Q. Через F обозначим вполне непрерывный оператор, определенный на S или на Q значения которого принадлежат X, а через U — непрерывную инволюцию сферы S в себя, без неподвижных точек при которой для почти всех натуральных n — образ $U(S^n)$ любой мерной сферы $S^n = E^{n+1}$ расположен в $E^{n+1} \cap S$, где E^{n+1} — $(n+1)$-мерное подпространство X.

Теорема 1. *Если отображение* f, *определенное на* Q *формулой* $f(x) = x - F(x)$ *удовлетворяет условию*

$$\frac{f(Ux)}{\|f(Ux)\|} \neq \frac{f(x)}{\|f(x)\|} \quad \text{для каждого} \quad x \in S, \tag{1}$$

то степень отображения f *является нечетной.*

Теорема 2. *Если отображение* f, *определенное на* Q *формулой* $f(x) = x - F(x)$ *удовлетворяет условию* (1), *то существует* $x_0 \in Q$ *такое, что* $F(x_0) = x_0$.

В случае когда U — антиподическое отображение, получаем отсюда результат Красносельского, обобщающий так наз. теорему об антиподах на пространства Банаха.

М. АЛЬТМАН, НЕПРЕРЫВНЫЕ ОТОБРАЖЕНИЯ ЛОКАЛЬНО ВЫПУКЛЫХ ПРОСТРАНСТВ НА СЕБЯ . . . . . стр. 37—39

В настоящей работе вводится понятие ϑ-отображения в узком смысле, являющееся обобщением понятия ε-отображения в узком смысле. Это обобщение дает возможность доказать теорему, являющуюся обобщением теоремы Борсука об ε-отображениях (в узком смысле), а также обобщений этой теоремы на пространства Банаха, полученных А. Гранасом.

А. КШИВИЦКИЙ, АППРОКСИМАЦИОННЫЙ МЕТОД РЕШЕНИЯ ПРОБЛЕМ РАССЕИВАНИЯ . . . . . стр. 41—45

В представленной работе предложен релятивистский аппроксимационный метод, который может оказаться особенно полезным для исследования процессов, где существенную роль играют сильные взаимодействия. В то время как в методах, применяемых в теории сильных взаимодействий ищем приближенных уравнений для амплитуд состояний, сущность нашего метода состоит в нахождении приближенных выражений для операторов взаимодействующих полей. Пользуемся возможностью непертурбационного разложения оператора поля в представлении Гейзенберга в ряд, члены которого пропорциональны нормальным произведениям асимптотических (для времени стремящегося к $\pm\infty$) операторов полей.

Г. ВАЛЕРЫСЬ, ЭМИССИОННЫЕ И АБСОРБЦИОННЫЕ СПЕКТРЫ МОЛЕКУЛ АКРИДИНОВОГО ЖЕЛТОГО В СОСТОЯНИИ ПРЕДВОЗБУЖДЕНИЯ . . . . . стр. 47—49

Исследованы спектры поглощения и испускания фосфоресценции и длительной флуоресценции акридинового желтого в желатине в зависимости от времени предвозбуждения. Оказалось, что параллельно с ростом продолжительности предвозбуждения возрастает поглощение и максимум полосы перемещается в направлении длинных волн. Этим изменениям сопутствует уменьшение интенсивности длительных флуоресценции и фосфоресценции, причем максимум полосы фосфоресценции перемещается в направлении коротких волн, тогда как положение максимума полосы длительной флуоресценции не подвергается изменению. Эти явления объясняются возниканием во время предвозбуждения непрочных изомеров красителя с положением электронных уровней, отличным от положения уровней в непредвозбужденной молекуле.